

# PHASE CONSTANTS IN THE FOCK-GONCHAROV QUANTIZATION OF CLUSTER VARIETIES

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**ABSTRACT.** A cluster variety of Fock and Goncharov is a scheme constructed from the data related to the cluster algebras of Fomin and Zelevinsky. A seed is a combinatorial data which can be encoded as an  $n \times n$  matrix with integer entries, or as a quiver in special cases, together with  $n$  formal variables. A mutation is a certain rule for transforming a seed into another seed; the new variables are related to the previous variables by some rational expressions. To each seed one attaches an  $n$ -dimensional torus, and by gluing the tori along the birational maps defined by the mutation formulas, one constructs a cluster variety. Quantization of a cluster variety assigns to each seed a non-commutative ring which deforms the classical ring of functions on the torus attached to the seed, as well as to each mutation an isomorphism of skew fields of fractions of these non-commutative rings. A representation realizes the non-commutative rings as algebras of operators on Hilbert spaces, and the quantum mutation isomorphisms as unitary maps between the Hilbert spaces that intertwine the operators for the rings. These unitary intertwiners are one of the major results of the Fock-Goncharov quantization of cluster varieties, and are given by the special function called the quantum dilogarithm. The classical mutations satisfy certain algebraic relations, which were known to be satisfied also by the corresponding intertwiners up to complex constants of modulus 1. The present paper shows by computation that these constants are all 1. One implication is that the mapping class group representations resulting from the application of the Fock-Goncharov quantization to the quantum Teichmüller theory are genuine, not projective.

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The present paper is a shortened version, prepared for journal submission, of the long version [K16l].

## 1. INTRODUCTION

A ‘cluster variety’ of Fock and Goncharov [FG09] [FG09b] can be viewed as a algebro-geometric space whose ring of regular functions is an ‘upper cluster algebra’ of Fomin and Zelevinsky [FZ02]. More precisely, it is a scheme over  $\mathbb{Z}$  constructed by gluing split algebraic tori  $(\mathbb{G}_m)^n$  along certain birational maps, where the tori are enumerated by the data called ‘seeds’ in the theory of cluster algebras, while the birational maps correspond to the ‘mutations’ of seeds. There are three different types of cluster varieties, namely  $\mathcal{A}$ ,  $\mathcal{X}$ , and  $\mathcal{D}$ , according to how these birational maps are given. In fact, only the cluster  $\mathcal{A}$ -variety provides a genuine example of a cluster algebra in the sense of [FZ02], while the  $\mathcal{X}$ - and the  $\mathcal{D}$ -varieties are certain generalizations.

Fix a ‘rank’  $n \in \mathbb{Z}_{>0}$ . Underlying a *seed* is a combinatorial data, which essentially can be encoded as an  $n \times n$  integer matrix  $\varepsilon$ , called the *exchange matrix*, which is required to be ‘skew-symmetrizable’, i.e. must be skew-symmetric when multiplied by some diagonal matrix from the left. In particular, when  $\varepsilon$  is skew-symmetric, it can be realized as the adjacency matrix of a ‘quiver’, i.e. a graph with oriented edges, without cycles of length 2. A seed is also equipped with  $n$  algebraically independent (formal) variables, which can be thought of as attached to the vertices of a quiver. If one picks any  $k \in \{1, \dots, n\}$ , or a vertex of a quiver, one can apply the *mutation*  $\mu_k$  to the seed to obtain another seed; the quiver or the exchange matrix changes according to a certain combinatorial rule, while the new  $n$  variables are related to the former  $n$  variables by certain rational formulas which depend on the exchange matrix and are ‘positive’, i.e. can be written as quotients of polynomials with positive coefficients. The mutation formulas for the exchange matrices and the attached variables might seem quite ad hoc at a first glance. So it is remarkable that such a structure appears in very many areas of mathematics.

What is even more remarkable is a Poisson-like structure on a cluster variety, which is written in a simple form on each torus in terms of the exchange matrix and is compatible under the mutation formulas. More precisely, the  $\mathcal{A}$ -, the  $\mathcal{X}$ -, and the  $\mathcal{D}$ -varieties are equipped with a 2-form (of ‘ $K_2$  class’), a Poisson structure, and a symplectic form, respectively. So it is sensible to investigate whether there exist ‘deformation quantizations’ of the  $\mathcal{X}$ - and the  $\mathcal{D}$ -varieties, which I explain now.

For a (classical) cluster variety we associate to each seed a torus  $(\mathbb{G}_m)^n$ , which can be viewed as being defined as the space whose ‘ring of regular functions’ is the ring of Laurent polynomials in  $n$  variables over  $\mathbb{Z}$ . So, one can think of this situation as attaching such a commutative ring to each seed. To each mutation we associate a birational map between the two tori, that is, an isomorphism of the fields of fractions of the attached Laurent polynomials rings.

To quantize a cluster variety means that we would like to associate to each seed a one-parameter family of non-commutative rings which deforms the classical commutative ring of Laurent polynomials, while to each mutation we associate an isomorphism of the skew fields of fractions of the non-commutative rings associated to the relevant seeds. First, this ‘quantum’ isomorphism of skew fields must be a deformation of the corresponding ‘classical’ isomorphism of fields of fractions of the commutative Laurent polynomial rings. Second, more importantly, there must be a ‘consistency’. Namely, there exists sequences of mutations that return the same exchange matrices after their application and whose induced classical isomorphisms of (commutative) fields of fractions are the identity maps. It is natural to require that the quantum isomorphisms of the skew fields of fractions induced by these sequences are also identity maps. However, such sequences are not classified in the classical setting. At this point one can only say that there are some known examples: the ‘rank 1 identity’  $\mu_k \circ \mu_k = \text{id}$  and ‘rank 2 identities’, including the ‘pentagon identity’  $\mu_i \mu_j \mu_i \mu_j \mu_i = (i\ j)$  in case  $\varepsilon_{ij} = -\varepsilon_{ji} \in \{1, -1\}$ , where  $(i\ j)$  stands for the relabelling  $i \leftrightarrow j$ . Fock and Goncharov [FG09] constructed quantum isomorphisms of skew fields associated to mutations and checked that they satisfy this consistency for these known sequences of mutations.

As is often the case for a ‘physical’ quantization, one would like to ‘represent’ the non-commutative ring associated to a seed as an algebra of operators, i.e. to find a representation of this ring on a Hilbert space, satisfying certain desirable analytic properties. Then, to each mutation one would like to associate a unitary map between the Hilbert spaces assigned to the relevant seeds that ‘intertwines’ the representations of the non-commutative rings which are related by the quantum isomorphism. In the Fock-Goncharov quantization [FG09], one advantage of having this intertwining map is that its formula is more transparent than the quantum isomorphism formulas for the non-commutative algebras. In particular, the intertwiner is written quite neatly in terms of a famous special

function called the ‘quantum dilogarithm’ of Faddeev and Kashaev [F95] [FK94], while the formula for the quantum isomorphism of algebras is not as enlightening.

A priori, the ‘consistency’ of these unitary intertwiners for mutations for the above mentioned sequences of mutations is not guaranteed, and must be verified separately if one wants to have it. However, Fock and Goncharov [FG09] deduce this consistency of intertwiners essentially by taking advantage of the ‘irreducibility’ of the representations of rings which are being intertwined. A rough idea is like the famous Schur’s lemma, which says that a vector space endomorphism that commutes with all the operators for an irreducible representation of a (finite) group on that vector space is a scalar operator. In particular, for each sequence of mutations inducing the identity classical isomorphism of fields of fractions, the composition of corresponding unitary intertwiners is a scalar operator on a relevant Hilbert space. One may interpret this result as having a unitary projective representation on a Hilbert space of the groupoid of mutations, where the word ‘projective’ carries the connotation that the relations are satisfied only up to complex scalars. As all operators are unitary, these scalars are of modulus 1, hence can be viewed as *phases*. The existence of these phase scalars, which can be thought of as a certain ‘anomaly’, is not at all a defect of the program of quantization of cluster varieties. As a matter of fact, this (projective) anomaly alone may already contain a very interesting bit of information. One of the prominent examples of cluster varieties are the *Teichmüller spaces* of Riemann surfaces, where the *mapping class groups* can be embedded into the groupoid of mutations. Quantization of cluster varieties is thus one approach to the ‘quantum Teichmüller theory’, one of the main results of which is the unitary projective representations of mapping class groups on Hilbert spaces. The anomaly can then be interpreted as group 2-cocycles of mapping class groups, or equivalently, central extensions of these groups, and this led to the works of Funar-Sergiescu [FS10] and Xu [X14].

However, these phase scalars for the projective anomaly have not been determined as precise complex numbers. The main result of the present paper is that these scalars are all 1 (Propositions 103, 105, 106, 110, 111). So, we get *genuine* representations of groupoid of mutations and mapping class groups on Hilbert spaces, instead of projective representations. To prove this I perform explicit computation involving unitary operators, instead of resorting just to the irreducibility of representations and the “Schur’s lemma”-type argument. While doing so I discovered an unfortunate mistake in [FG09], which is not merely a typo but is easily fixable; namely, their intertwiner does not intertwine the representation they take in that paper for the non-commutative ring associated to each seed (Lem.87). One should replace it by their old representation which was used in a previous paper [FG07]; one possible explanation for a reason why their old representation is more favorable than their new one is in terms of the ‘canonical’ quantization of cotangent bundles, which may be discussed in the future work [KS]. In addition, I replace their ‘Heisenberg relations’ by the corresponding ‘Weyl relations’ which are more rigid, and also introduce and study the concept of *special affine shift operators* on  $L^2(\mathbb{R}^n)$  (§5.5), which are induced by the ‘special affine’ transformations  $SL_{\pm}(n, \mathbb{R}) \ltimes \mathbb{R}^n$  of  $\mathbb{R}^n$ , in order to facilitate notations and computations for unitary operators, which may be adapted for future projects too. A rather easy yet crucial observation is that any special affine shift operators that is proportional to the identity operator is exactly the identity operator (Lem.75). The key point of the proof of the main result of the present paper is to show, using quantum dilogarithm identities, that the composition of the unitary intertwiners corresponding to the relevant mutation sequences for each ‘consistency’ relation is a special affine shift operator. As Fock and Goncharov proved that these are scalar operators, it follows that these are the identity operators. Among the quantum dilogarithm identities is the famous pentagon identity of the quantum dilogarithm function. Certain generalizations of the pentagon identity are observed, namely the ‘hexagon’ and the ‘octagon’ identities (§6.6), which I claim to follow from the pentagon; these identities will be treated more thoroughly in [KY].

The initial version of the present paper contains more details and rigorous proofs regarding the functional analysis that is used in the construction of the quantization of cluster varieties, as well as more friendly discussions on various aspects of the quantization. It is written in such a manner to help any reader, including myself, who is interested in understanding and eventually getting hands on the work of Fock and Goncharov, with the expense of being pretty long. In particular, it will serve as a basic framework for a joint work in progress with Carlos Scarinci on quantization of moduli spaces of 3-dimensional globally hyperbolic spacetimes [KS]. The present article is a shortened version of it prepared for journal submission, focusing on the original result, namely the computation of the phase constants appearing in the unitary representation of Fock-Goncharov’s quantum cluster varieties.

The triviality of these phases for example lets us remove the multiplicative constants in the quantum dilogarithm identities as stated in [KN11, Thm.4.6]. But at the same time it unfortunately deprives of the representation theoretic meaning from the works [FS10] [X14] at the moment, leaving only the group theoretic meaning, which is further investigated in my work [K12] in comparison with another quantization of Teichmüller spaces obtained

by Kashaev. It was observed in [K12] that these two quantizations of Teichmüller spaces are different in a very interesting sense, related to braid groups. In my opinion, this suggests that there exists a quantization of cluster varieties 1) with non-trivial phase constants, and furthermore 2) with these constants being (half-)integer powers of  $\zeta = e^{-\frac{\pi\sqrt{-1}}{6}}c_h$ , where  $c_h = e^{-\frac{\pi\sqrt{-1}}{12}(h+h^{-1})}$  (5.46) and  $\hbar$  is the real quantization parameter. However, such a quantization has not been constructed yet, hence calls for a future investigation.

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## 2. CLUSTER $\mathcal{A}$ -, $\mathcal{X}$ - AND $\mathcal{D}$ -VARIETIES

I shall describe Fock-Goncharov's cluster  $\mathcal{A}$ -varieties, cluster  $\mathcal{X}$ -varieties, and cluster  $\mathcal{D}$ -varieties. Definitions and treatment in this section are from [FG09] and references therein, with certain modifications.

**2.1. Seed and seed tori.** In the present paper, we choose and fix one positive integer  $n$ , which can be regarded as the ‘rank’ of the relevant cluster algebras/varieties.

**Definition 1.** Let  $K$  be a field. We say that an  $n \times n$  matrix  $\varepsilon = (\varepsilon_{ij})_{i,j=1,\dots,n}$  with entries in  $K$  is skew-symmetrizable if there exist  $d_1, \dots, d_n \in K^*$  such that the matrix  $(\widehat{\varepsilon}_{ij})_{i,j}$  defined as

$$(2.1) \quad \widehat{\varepsilon}_{ij} := \varepsilon_{ij} d_j^{-1}$$

is skew-symmetric, i.e.  $\widehat{\varepsilon}_{ij} = -\widehat{\varepsilon}_{ji}$ ,  $\forall i, j$ . We call such a collection  $d = (d_i)_{i=1,\dots,n}$  a skew-symmetrizer of the matrix  $(\varepsilon_{ij})_{i,j}$ .

For any skew-symmetrizable matrix over  $K = \mathbb{Q}$ , one can always find a skew-symmetrizer consisting only of integers.

**Definition 2** (seed). An  $\mathcal{A}$ -seed  $\Gamma$  is a triple  $(\varepsilon, d, \{A_i\}_{i=1}^n)$  of an  $n \times n$  skew-symmetrizable integer matrix  $\varepsilon$ , a skew-symmetrizer  $d$  consisting of positive integers, and an ordered set of formal variables  $A_1, A_2, \dots, A_n$  called the cluster  $\mathcal{A}$ -variables.

An  $\mathcal{X}$ -seed  $\Gamma$  is a triple  $(\varepsilon, d, \{X_i\}_{i=1}^n)$  of  $\varepsilon, d$  as above, and an ordered set of variables  $X_1, \dots, X_n$  called the cluster  $\mathcal{X}$ -variables.

An  $\mathcal{D}$ -seed  $\Gamma$  is a triple  $(\varepsilon, d, \{B_i, X_i\}_{i=1}^n)$  of  $\varepsilon, d$  as above, and an ordered set of variables  $B_1, \dots, B_n, X_1, \dots, X_n$  called the cluster  $\mathcal{D}$ -variables.

We call any of these a seed,  $\varepsilon$  the exchange matrix of the seed, the relevant formal variables the cluster variables of the seed, and the relevant symbols  $\mathcal{A}, \mathcal{X}, \mathcal{D}$  the kind of the seed. We write a seed by  $\Gamma = (\varepsilon, d, *)$  if it is clear from the context what the cluster variables are.

Some words must be put in order. What Fock and Goncharov [FG09] call a ‘feed’ is essentially a seed without the cluster variables, i.e. the exchange matrix  $\varepsilon$  and the skew-symmetrizer  $d$ . I think it makes more sense to use seeds as defined in Def.2 instead of feeds, in the construction of ‘cluster varieties’, whence I do so in the present paper. An  $\mathcal{A}$ -seed, with the data of  $d$  left out, is what is called a ‘labeled seed’ with ‘trivial coefficients’ in the theory of cluster algebras initiated by Fomin and Zelevinsky [FZ02]. A  $\mathcal{D}$ -seed minus the data of  $d$  is what is called a ‘labeled seed’ with ‘coefficients’ in the ‘universal semifield’ in [FZ07];  $\{B_i\}_{i=1}^n$  is their ‘cluster’ and  $X_i$ ’s their coefficients. Readers will be able to verify these comparison when they get to the formulas for the ‘mutations’ in §2.2. On the other hand, the cluster  $\mathcal{X}$ -variables are not the usual ‘cluster variables’ in the sense of Fomin-Zelevinsky. So, a ‘seed’ as defined in the above Def.2 is, say, a ‘generalized seed together with the choice of a skew-symmetrizer’.

For a  $\mathcal{D}$ -seed there is an interesting set of redundant variables, which are included in what we will casually refer to as cluster ( $\mathcal{D}$ -)variables:

**Definition 3** (tilde variables for  $\mathcal{D}$ -seed). For a  $\mathcal{D}$ -seed  $\Gamma = (\varepsilon, d, \{B_i, X_i\}_{i=1}^n)$ , define new variables  $\tilde{X}_i$ ,  $i = 1, \dots, n$ , by

$$\tilde{X}_i := X_i \prod_{j=1}^n B_j^{\varepsilon_{ij}}, \quad \forall i = 1, \dots, n.$$

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not a standard terminology

Fock and Goncharov call these ‘cluster coordinates’ in [FG09].

This symbol  $\Gamma$  has nothing to do with the cluster mapping class group (cluster modular group) in [FG09].

Goncharov says that the uncommon terminology ‘feed’ used in [FG09] was supposed to be a joke.

For each seed, we consider a parametrized topological space whose coordinate functions are identified with the cluster variables.

**Definition 4** (seed tori). *To the seeds in Def.2, all denoted by  $\Gamma$  by abuse of notations, we assign respectively the split algebraic tori*

$$\mathcal{A}_\Gamma := (\mathbb{G}_m)^n, \quad \mathcal{X}_\Gamma = (\mathbb{G}_m)^n, \quad \mathcal{D}_\Gamma = (\mathbb{G}_m)^{2n},$$

*called the seed  $\mathcal{A}$ -torus, the seed  $\mathcal{X}$ -torus, and the seed  $\mathcal{D}$ -torus, where  $\mathbb{G}_m$  is the multiplicative algebraic group, defined as  $\mathbb{G}_m(K) = K^*$  for any field  $K$ . We call these seed tori, collectively. We identify the canonical coordinate functions of  $\mathcal{A}_\Gamma$  with the seed's cluster variables  $A_i$ 's, those of  $\mathcal{X}_\Gamma$  with  $X_i$ 's, and those of  $\mathcal{D}_\Gamma$  with  $B_i$ 's and  $X_i$ 's.*

So, one  $\mathcal{A}$ -seed  $\Gamma$  gives us one topological space  $\mathcal{A}_\Gamma$ . We shall consider  $\mathcal{A}_\Gamma$  for different  $\Gamma$ 's, and glue them together by certain algebraic formulas, to construct a 'cluster  $\mathcal{A}$ -variety'. We do likewise for the spaces  $\mathcal{X}_\Gamma$  for different  $\Gamma$ 's to construct a 'cluster  $\mathcal{X}$ -variety', and similarly for  $\mathcal{D}_\Gamma$ 's to construct a 'cluster  $\mathcal{D}$ -variety'. I will explain these gluings in the subsequent subsections. Note that we never glue seed tori of different kinds, whence the abuse of notations committed by labeling the different kinds of seeds by the same letter  $\Gamma$  will not be too harmful.

We construct a local patch as an irreducible affine variety defined as the Spec of the specified *ring of regular functions* on it. This ring of regular functions is what we care more than the topology of the local patch. We take this ring to be the ring of Laurent polynomials in  $n$  (resp.  $2n$ ) variables over  $\mathbb{Z}$  (instead of, say, the ring of polynomials in  $n$  variables). The corresponding affine variety, i.e. the Spec of this ring, is the split algebraic torus of rank  $n$  (or  $2n$ ), showing why it is sensible to have Def.4.

What make the cluster varieties a lot richer and more interesting are the remarkable geometric structures they possess on top of their topology, which are defined on their local patches as follows.

**Definition 5** (geometric structures on seed tori). *We equip the following 2-form on  $\mathcal{A}_\Gamma$*

$$(2.2) \quad \Omega_\Gamma = \sum_{i,j \in \{1, \dots, n\}} \tilde{\varepsilon}_{ij} d \log A_i \wedge d \log A_j, \quad \text{where } \tilde{\varepsilon}_{ij} := d_i \varepsilon_{ij},$$

*and the following Poisson structure on  $\mathcal{X}_\Gamma$*

$$\{X_i, X_j\} = \hat{\varepsilon}_{ij} X_i X_j, \quad \forall i, j \in \{1, \dots, n\}.$$

*On  $\mathcal{D}_\Gamma$  we consider the Poisson structure defined by*

$$(2.3) \quad \{B_i, B_j\} = 0, \quad \{X_i, B_j\} = d_i^{-1} \delta_{ij} X_i B_j, \quad \{X_i, X_j\} = \hat{\varepsilon}_{ij} X_i X_j,$$

*where  $\delta_{ij}$  is the Kronecker delta, as well as the following 2-form*

$$(2.4) \quad -\frac{1}{2} \sum_{i,j \in \{1, \dots, n\}} \tilde{\varepsilon}_{ij} d \log B_i \wedge d \log B_j - \sum_{i \in \{1, \dots, n\}} d_i d \log B_i \wedge d \log X_i.$$

**Remark 6.** *Fock and Goncharov say that  $\Omega_\Gamma$  on  $\mathcal{A}_\Gamma$  is of 'K<sub>2</sub> class'.*

**Lemma 7** ([FG09]). *The 2-form (2.4) on  $\mathcal{D}_\Gamma$  is a symplectic form, and is compatible with the Poisson structure (2.3) on it.*

**2.2. Seed mutations and seed automorphisms.** In order to study the gluing of the seed tori, we first need to investigate the following transformation rules for seeds, which we can also understand as a way of recursively creating new seeds from previously constructed ones.

**Definition 8** (seed mutation). *For  $k \in \{1, 2, \dots, n\}$ , a seed  $\Gamma' = (\varepsilon', d', *)$  is said to be obtained by applying the seed mutation  $\mu_k$  in the direction  $k$  to a seed  $\Gamma = (\varepsilon, d, *)$ , if all of the following hold:*

- *the exchange matrices are related by*

$$(2.5) \quad \varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij} + \frac{1}{2}(|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|) & \text{otherwise.} \end{cases}$$

- *the skew symmetrizers are related by*

$$(2.6) \quad d'_i = d_i, \quad \forall i.$$

- *the cluster variables are related by:*

1) In the case of  $\mathcal{A}$ -seeds:

$$(2.7) \quad A'_i = \begin{cases} A_i & \text{if } i \neq k, \\ A_k^{-1} \left( \prod_{j \mid \varepsilon_{kj} > 0} A_j^{\varepsilon_{kj}} + \prod_{j \mid \varepsilon_{kj} < 0} A_j^{-\varepsilon_{kj}} \right) & \text{if } i = k, \end{cases}$$

where  $\Gamma = (\varepsilon, d, \{A_i\}_{i=1}^n)$  and  $\Gamma' = (\varepsilon', d', \{A'_i\}_{i=1}^n)$ .

2) In the case of  $\mathcal{X}$ -seeds:

$$(2.8) \quad X'_i = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i \left( 1 + X_k^{\text{sgn}(-\varepsilon_{ik})} \right)^{-\varepsilon_{ik}} & \text{if } i \neq k, \end{cases}$$

where  $\Gamma = (\varepsilon, d, \{X_i\}_{i=1}^n)$  and  $\Gamma' = (\varepsilon', d', \{X'_i\}_{i=1}^n)$ , and

$$\text{sgn}(a) = \begin{cases} 1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0. \end{cases}$$

3) In the case of  $\mathcal{D}$ -seeds: the formulas in (2.8), together with

$$(2.9) \quad B'_i = \begin{cases} B_i & \text{if } i \neq k, \\ \frac{\left( \prod_{j \mid \varepsilon_{kj} < 0} B_j^{-\varepsilon_{kj}} \right) + X_k \left( \prod_{j \mid \varepsilon_{kj} > 0} B_j^{\varepsilon_{kj}} \right)}{B_k(1 + X_k)} & \text{if } i = k, \end{cases}$$

where  $\Gamma = (\varepsilon, d, \{B_i, X_i\}_{i=1}^n)$  and  $\Gamma' = (\varepsilon', d', \{B'_i, X'_i\}_{i=1}^n)$ .

We denote such a situation by  $\mu_k(\Gamma) = \Gamma'$ , or  $\Gamma \xrightarrow{k} \Gamma'$ . We call this procedure a mutation.

The new cluster variables are viewed as elements of the ‘ambient field’, the field of all rational functions in the previous cluster variables over  $\mathbb{Q}$ . Each of (2.7)–(2.9) should be thought of as an equality in the ambient field.

**Remark 9.** [FG09] uses the notations  $\mathbb{B}_k^+ := \prod_{j \mid \varepsilon_{kj} > 0} B_j^{\varepsilon_{kj}}$  and  $\mathbb{B}_k^- := \prod_{j \mid \varepsilon_{kj} < 0} B_j^{-\varepsilon_{kj}}$ , which help simplifying the formula (2.9).

**Definition 10** (seed automorphism). A seed  $\Gamma' = (\varepsilon', d', *)$  is said to be obtained by applying the seed automorphism  $P_\sigma$  to a seed  $\Gamma = (\varepsilon, d, *)$  for a permutation  $\sigma$  of  $\{1, 2, \dots, n\}$ , if

$$d'_{\sigma(i)} = d_i, \quad \varepsilon'_{\sigma(i)\sigma(j)} = \varepsilon_{ij}, \quad \forall i, j,$$

and

$$A'_{\sigma(i)} = A_i, \quad (\text{for } \mathcal{A}\text{-seeds}), \quad X'_{\sigma(i)} = X_i, \quad (\text{for } \mathcal{X}\text{-, } \mathcal{D}\text{-seeds}), \quad B'_{\sigma(i)} = B_i, \quad (\text{for } \mathcal{D}\text{-seeds}),$$

for all  $i = 1, 2, \dots, n$ . We denote such a situation by  $P_\sigma(\Gamma) = \Gamma'$ .

We can consider the mutations and seed automorphisms as being applied to seeds from the left, and use the usual notation  $\circ$  for the composition of them. So, when we apply a finite sequence of mutations and permutations to a seed, we read the sequence from right to left.

**Definition 11** (cluster transformations). A cluster transformation is a finite sequence of mutations and seed automorphisms, together with a seed  $\Gamma$  on which the sequence is to be applied to. If the resulting seed is  $\Gamma'$ , we say that this cluster transformation connects  $\Gamma$  to  $\Gamma'$ . We call mutations and seed automorphisms elementary cluster transformations.

Note that, each of the symbols  $\mu_k$  and  $P_\sigma$  stands for many different elementary cluster transformations, for it can be thought of as being applied to different seeds, which could be of any of the three kinds.

**2.3. Cluster modular groupoids.** In practice, we start from a single seed, which we often call an ‘initial seed’, and produce new seeds by applying cluster transformations to it. Then the set of seeds thus created will be in bijection with the set of all finite sequences of mutations and permutations. However, I would like to identify two seeds whenever they are ‘essentially’ the same, in the sense I explain now.

Given any cluster transformation, say, connecting a seed  $\Gamma$  to a seed  $\Gamma'$ , one obtains a well-defined identification of the cluster variables for  $\Gamma'$  as rational functions in the cluster variables for  $\Gamma$ , by ‘composing’ the formulas in Definitions 8 and 10.

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$P_\sigma$  is denoted simply by  $\sigma$  in [FG09].

**Definition 12** (trivial cluster transformations). A cluster transformation connecting a seed  $\Gamma = (\varepsilon, d, *)$  to a seed  $\Gamma' = (\varepsilon', d', *)$  is said to be weakly trivial if  $\varepsilon' = \varepsilon$  and  $d' = d$ . A cluster transformation is said to be  $\mathcal{A}$ -trivial if it is weakly trivial and induces the identity map between the cluster  $\mathcal{A}$ -variables, i.e. the  $i$ -th cluster  $\mathcal{A}$ -variable  $A_i$  for  $\Gamma$  equals the  $i$ -th cluster  $\mathcal{A}$ -variable  $A'_i$  for  $\Gamma'$ . The notions  $\mathcal{X}$ -trivial and  $\mathcal{D}$ -trivial are defined analogously. A cluster transformation is said to be trivial if it is  $\mathcal{A}$ -trivial,  $\mathcal{X}$ -trivial, or  $\mathcal{D}$ -trivial.

**Remark 13.** In [FG09] a ‘(feed) cluster transformation’ means a sequence of mutations and permutations applied to a feed  $(\varepsilon, d)$ , and it is said to be ‘trivial’ if it is  $\mathcal{A}$ -trivial and  $\mathcal{X}$ -trivial at the same time, when applied to  $\mathcal{A}$ -seeds and  $\mathcal{X}$ -seeds whose underlying feeds are  $(\varepsilon, d)$ .

**Definition 14** (identification of seeds). If a cluster transformation connecting an  $\mathcal{A}$ -seed  $\Gamma$  to an  $\mathcal{A}$ -seed  $\Gamma'$  is  $\mathcal{A}$ -trivial, we identify  $\Gamma$  and  $\Gamma'$  as  $\mathcal{A}$ -seeds, and write  $\Gamma = \Gamma'$ . Likewise for  $\mathcal{X}$ -seeds and  $\mathcal{D}$ -seeds.

Keeping this identification of seeds in mind, we consider:

**Definition 15** (equivalence of seeds). The two seeds are equivalent if they are connected by a cluster transformation. For a seed  $\Gamma$ , denote by  $\mathcal{C} = |\Gamma|$  the equivalence class of  $\Gamma$ .

Let us now investigate some examples of trivial cluster transformations. It is a standard and straightforward exercise to show that the following example is indeed a trivial cluster transformation.

**Lemma 16** (involution identity of a mutation).  $\mu_k \circ \mu_k$  is a trivial cluster transformation on any seed of any kind. That is, if we write  $\Gamma' = \mu_k(\Gamma)$  and  $\Gamma'' = \mu_k(\Gamma')$ , with  $\Gamma'' = (\varepsilon'', d'', *)$  and  $\Gamma = (\varepsilon, d, *)$ , then  $\varepsilon'' = \varepsilon$ ,  $d'' = d$ , and  $A''_i = A_i$ ,  $\forall i$  (for  $\mathcal{A}$ -seeds),  $X''_i = X'_i$ ,  $\forall i$  (for  $\mathcal{X}$ - and  $\mathcal{D}$ -seeds), and  $B''_i = B_i$ ,  $\forall i$  (for  $\mathcal{D}$ -seeds). ■

In particular, we identify  $\mu_k(\mu_k(\Gamma))$  and  $\Gamma$ . Some more easy-to-see trivial cluster transformations involving the seed automorphisms are as follows, which I also omit the proof of.

**Lemma 17** (permutation identities). For any permutations  $\sigma, \gamma$  of  $\{1, \dots, n\}$  and any  $k \in \{1, \dots, n\}$ ,

$$P_\sigma \circ P_\gamma = P_{\sigma \circ \gamma}, \quad P_\sigma \circ \mu_k \circ P_{\sigma^{-1}} = \mu_{\sigma(k)}, \quad P_{\text{id}} = \text{id},$$

all of which hold as identities when applied to any seed of any kind, where  $\text{id}$  stands either for the identity permutation or the identity cluster transformation. ■

The first two identities in this statement can be translated into saying that  $P_\sigma \circ P_\gamma \circ P_{\sigma \circ \gamma}^{-1}$  and  $P_\sigma \circ \mu_k \circ P_{\sigma^{-1}} \circ \mu_{\sigma(k)}^{-1}$  are trivial cluster transformations on any seed of any kind. Besides these, there remain certain interesting trivial cluster transformations, pointed out e.g. in eq.(20) in of [FG09, p.238]:

**Lemma 18** (The  $(h+2)$ -gon relations). Suppose that a seed  $\Gamma = (\varepsilon, d, *)$  of any kind satisfies

$$\varepsilon_{ij} = -p\varepsilon_{ji} \quad \text{and} \quad |\varepsilon_{ij}| = p$$

for some  $i, j \in \{1, \dots, n\}$  and  $p \in \{0, 1, 2, 3\}$ . Let  $h = 2, 3, 4, 6$  for  $p = 0, 1, 2, 3$ , respectively. Denote by  $(i\ j)$  the transposition permutation of  $\{1, \dots, n\}$  interchanging  $i$  and  $j$ . Then

$$(2.10) \quad (P_{(i\ j)} \circ \mu_i)^{h+2} \text{ applied to } \Gamma \text{ is a trivial cluster transformation.}$$

The permutation identities in Lem.17 do not really require a proof. The involution identity in Lem.16 can be thought of as coming from ‘rank one’ cluster algebras, namely, of Dynkin type  $A_1$ . The more serious identities in Lem.18 can be thought of as coming from ‘rank two’ cluster algebras of Dynkin types  $A_1 \times A_1$ ,  $A_2$ ,  $B_2$ ,  $G_2$ , respectively, as pointed out in [FG09]. Fock and Goncharov say in [FG09] that they do not know how to find more trivial cluster transformations that are not consequences of the ones already discussed, let alone how to describe the complete list of trivial cluster transformations, which certainly exists. Thus they take only these known relations to formulate the problem and the result of their quantization. More discussion on trivial cluster transformations will appear in a joint work in progress with Masahito Yamazaki [KY].

An idealistic structure that the equivalent seeds form is the following:

**Definition 19.** The  $\mathcal{A}$ -cluster modular groupoid  $\mathcal{G}^{\mathcal{A}} = \mathcal{G}_{\mathcal{C}}^{\mathcal{A}}$  associated to an equivalence class  $\mathcal{C}$  of  $\mathcal{A}$ -seeds is a category whose set of objects is  $\mathcal{C}$ , and whose set of morphisms  $\text{Hom}_{\mathcal{G}^{\mathcal{A}}}(\Gamma, \Gamma')$  from an object  $\Gamma$  to  $\Gamma'$  is the set of all cluster transformations from  $\Gamma$  to  $\Gamma'$  modulo  $\mathcal{A}$ -trivial cluster transformations.

Likewise for the  $\mathcal{X}$ -cluster modular groupoid  $\mathcal{G}^{\mathcal{X}}$  and the  $\mathcal{D}$ -cluster modular groupoid  $\mathcal{G}^{\mathcal{D}}$ .

Identifying seeds connected by seed automorphisms amounts to considering Fomin-Zelevinsky’s notion of ‘unlabeled’ seeds.

[FG09] restricts only to  $\mathcal{A}$ -seeds and  $\mathcal{X}$ -seeds, but I think it also holds for  $\mathcal{D}$ -seeds.

[FG09] uses  $\varepsilon_{ij} = -p\varepsilon_{ji} = -p$ , but it doesn’t matter.

Here ‘modulo’ means the following. Consider cluster transformations  $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ , where some of them may be empty sequences of elementary cluster transformations. If  $\mathbf{m}_2$  is a  $\mathcal{A}$ -trivial cluster transformation, then  $\mathbf{m}_1 \circ \mathbf{m}_2 \circ \mathbf{m}_3$  and  $\mathbf{m}_1 \circ \mathbf{m}_3$  are viewed as the same morphism in the category  $\mathcal{G}^{\mathcal{A}}$ . Likewise for  $\mathcal{G}^{\mathcal{X}}$  and  $\mathcal{G}^{\mathcal{D}}$ . I myself have always found the following description of these groupoids useful.

**Lemma 20.** *The set of morphisms  $\text{Hom}_{\mathcal{G}^{\mathcal{A}}}(\Gamma, \Gamma')$  between any two objects of  $\mathcal{G}^{\mathcal{A}} = \mathcal{G}_{\mathcal{C}}^{\mathcal{A}}$  consists of exactly one element. Likewise for  $\mathcal{G}^{\mathcal{X}}$  and  $\mathcal{G}^{\mathcal{D}}$ .*

*Proof.* By definition of  $\mathcal{C}$ , the set  $\text{Hom}_{\mathcal{G}^{\mathcal{A}}}(\Gamma, \Gamma')$  has at least one element. Suppose  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , which are sequences of mutations and permutations being applied to  $\Gamma$ , are its elements. Then  $\mathbf{m}_1(\Gamma) = \Gamma' = \mathbf{m}_2(\Gamma)$ . Let  $\mathbf{m}_2^{-1}$  be the sequence obtained by inverting each entry of the sequence  $\mathbf{m}_2$  and putting in the reverse order, being applied to  $\Gamma'$ ; here, ‘inverting’ means replacing each  $\mu_i$  and  $P_\sigma$  by  $\mu_i$  and  $P_{\sigma^{-1}}$  respectively. Then, a repeated application of Lemmas 16 and 17 tells us  $(\mathbf{m}_2^{-1} \circ \mathbf{m}_2)(\Gamma) = \Gamma$ , hence  $\mathbf{m}_2^{-1}(\Gamma') = \Gamma$ . Define cluster transformations  $\mathbf{m}_3 := \mathbf{m}_2^{-1} \circ \mathbf{m}_1$ , being applied to  $\Gamma$ , and  $\mathbf{m}_4 := \mathbf{m}_2 \circ \mathbf{m}_2^{-1}$ , being applied to  $\Gamma'$ . Then  $\mathbf{m}_3(\Gamma) = \mathbf{m}_2^{-1}(\Gamma) = \Gamma$  and  $\mathbf{m}_4(\Gamma') = \mathbf{m}_2(\Gamma) = \Gamma'$ , hence  $\mathbf{m}_3$  and  $\mathbf{m}_4$  are trivial cluster transformations. Thus  $\mathbf{m}_2$  and  $\mathbf{m}_2 \circ \mathbf{m}_3$  are the same morphisms in  $\mathcal{G}^{\mathcal{A}}$  from  $\Gamma$  to  $\Gamma'$ . Note  $\mathbf{m}_2 \circ \mathbf{m}_3 = \mathbf{m}_2 \circ \mathbf{m}_2^{-1} \circ \mathbf{m}_1 = \mathbf{m}_4 \circ \mathbf{m}_1$ , which is the same morphism as  $\mathbf{m}_1$ . So  $\mathbf{m}_2$  is the same morphism as  $\mathbf{m}_1$ , which is the desired result for  $\mathcal{G}^{\mathcal{A}}$ . Similarly for  $\mathcal{G}^{\mathcal{X}}$  and  $\mathcal{G}^{\mathcal{D}}$ .  $\blacksquare$

A slightly less idealistic version is the following, taking into account only the known relations.

**Definition 21.** *The saturated  $\mathcal{A}$ -cluster modular groupoid  $\widehat{\mathcal{G}}^{\mathcal{A}} = \widehat{\mathcal{G}}_{\mathcal{C}}^{\mathcal{A}}$  associated to an equivalence class  $\mathcal{C}$  of  $\mathcal{A}$ -seeds is a category whose set of objects is  $\mathcal{C}$ , and whose set of morphisms  $\text{Hom}_{\widehat{\mathcal{G}}^{\mathcal{A}}}(\Gamma, \Gamma')$  from an object  $\Gamma$  to  $\Gamma'$  is the set of all cluster transformations from  $\Gamma$  to  $\Gamma'$  modulo only the trivial cluster transformations that are described in Lemmas 16, 17 and 18. Likewise for  $\widehat{\mathcal{G}}^{\mathcal{X}}$  and  $\widehat{\mathcal{G}}^{\mathcal{D}}$ .*

**Definition 22.** *In these groupoids, an elementary morphism is a morphism representing an elementary cluster transformation.*

As we shall soon see, these groupoids provide a handy way to formulate the construction of ‘cluster varieties’ and the quantization. However, Fock and Goncharov [FG09] define groupoids based on ‘feeds’  $(\varepsilon, d)$ , rather than seeds. Using theirs has an advantage as it allows to define the notion of (saturated) cluster modular group, or (saturated) cluster mapping class group, as the group of automorphisms of one feed; in particular, upon quantization, one gets projective representations of the cluster mapping class group. However, in view of the properties of the scheme that we will soon construct, feeds do not seem to be the best way to go.

**2.4. Cluster varieties.** Each elementary cluster transformation, say from a seed  $\Gamma$  to a seed  $\Gamma'$ , induces a rational map from the seed torus for  $\Gamma$  to that for  $\Gamma'$ , denoted by the same symbol as the elementary cluster transformation itself, defined in the level of functions by the identification formulas for the cluster variables, obtained by ‘composing’ the formulas in Definitions 8 and 10 as mentioned earlier. For example, we define the rational map  $\mu_k : \mathcal{X}_\Gamma \rightarrow \mathcal{X}_{\Gamma'}$  between the seed  $\mathcal{X}$ -tori by describing what the pullbacks of the coordinate functions of the torus  $\mathcal{X}_{\Gamma'}$  are, in terms of coordinate functions of  $\mathcal{X}_\Gamma$ :

$$\mu_k^*(X'_i) = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i \left(1 + X_k^{\text{sgn}(-\varepsilon_{ik})}\right)^{-\varepsilon_{ik}} & \text{if } i \neq k, \end{cases}$$

and  $\mu_k^*(X_i'^{-1}) = \mu_k^*(X_i')^{-1}$ ,  $\forall i$ .

It is easy to see that the map  $P_\sigma$  between two seed tori is an isomorphism of varieties. The map  $\mu_k$  between two seed tori is indeed a rational map, for it is defined on the subsets of the respective seed tori on which the denominators of the transformation formulas are not zero, which are open subsets whose complements in the tori are of lower dimension. As each  $\mu_k$  has a rational inverse map, namely  $\mu_k$  (Lem.16), we see that  $\mu_k$  is in fact a birational map.

So we can now say that any cluster transformation from a seed  $\Gamma$  to a seed  $\Gamma'$  induces a birational map between the two seed tori. We finally construct cluster varieties by gluing the seed tori along these birational maps.

**Definition 23** (cluster varieties). *The cluster  $\mathcal{A}$ -variety for an equivalence class  $\mathcal{C}$  of  $\mathcal{A}$ -seeds, is a scheme obtained by gluing all the seed tori  $\mathcal{A}_\Gamma$  for  $\Gamma$  in this equivalence class  $\mathcal{C}$  using the birational maps associated to cluster transformations. We denote it by  $\mathcal{A}_{\mathcal{C}}$ , or just  $\mathcal{A}$  if the equivalence class is apparent from the context.*

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probably not a completely standard terminology (used by Funar, Sergiescu and collaborators)

in the case of cluster variety ‘coming from a surface’, this group coincides with the ‘mapping class group’ of the surface.



Define the cluster  $\mathcal{X}$ -variety  $\mathcal{X} = \mathcal{X}_{\mathcal{C}}$  and the cluster  $\mathcal{D}$ -variety  $\mathcal{D} = \mathcal{D}_{\mathcal{C}}$  in a similar fashion.

In order for  $\mathcal{A}_{\mathcal{C}}$  to be well-defined, for each pair of seeds  $(\Gamma, \Gamma')$  in  $\mathcal{C}$  we must have a unique birational map  $\mathcal{A}_{\Gamma} \rightarrow \mathcal{A}_{\Gamma'}$  along which we glue the two tori. There may be many different sequences of mutations and permutations that connect  $\Gamma$  to  $\Gamma'$ , and we should make sure that they induce the same birational map on the tori. Readers can easily deduce this by using Lem.20, using the fact that trivial cluster transformations induce identity maps between the tori. Similarly for  $\mathcal{X}_{\mathcal{C}}$  and  $\mathcal{D}_{\mathcal{C}}$  too.

This definition is different from that of Fock and Goncharov [FG09]. Their cluster variety is defined as the Spec of the ring of global regular functions on the scheme constructed in a similar way as in Def.23 using ‘feed tori’ instead of seed tori. In our seed-version, this ring could be defined as follows.

**Definition 24.** For a seed  $\Gamma$  of any kind, denote by  $R_{\Gamma}$  the ring of regular functions on the seed torus for  $\Gamma$ , and let  $\mathbb{Q}_{\Gamma}$  be the field of fractions of  $R_{\Gamma}$ .

For equivalent seeds  $\Gamma$  and  $\Gamma'$ , denote by  $\mu_{\Gamma, \Gamma'}^*$  the unique isomorphism  $\mathbb{Q}_{\Gamma'} \rightarrow \mathbb{Q}_{\Gamma}$  of fields as described above on the generators.

For an equivalence class  $\mathcal{C}$  of seeds, the ring of global regular functions  $\mathbb{L}_{\mathcal{C}}$  is defined as the following subring of  $\mathbb{Q}_{\Gamma}$

$$(2.11) \quad \mathbb{L}_{\Gamma} := \bigcap_{\Gamma' \in \mathcal{C}} \mu_{\Gamma, \Gamma'}^*(R_{\Gamma'}) \subset \mathbb{Q}_{\Gamma},$$

for any chosen  $\Gamma \in \mathcal{C}$ .

To be a bit more explicit about the above notation, the ring  $R_{\Gamma}$  for an  $\mathcal{A}$ -seed  $\Gamma = (\varepsilon, d, \{A_i\}_{i=1}^n)$ , for example, is the ring  $\mathbb{Z}[A_1, A_1^{-1}, \dots, A_n, A_n^{-1}]$ . Note that (2.11) ‘does not depend on the choice of  $\Gamma \in \mathcal{C}$ ’ in the sense that

$$\mathbb{L}_{\Gamma} = \mu_{\Gamma, \Gamma'}^*(\mathbb{L}_{\Gamma'}), \quad \forall \Gamma, \Gamma' \in \mathcal{C}.$$

We collectively denote all  $\mathbb{L}_{\Gamma}$  for  $\Gamma \in \mathcal{C}$  by the symbol  $\mathbb{L}_{\mathcal{C}}$ .

**Remark 25.** As  $R_{\Gamma}$  is the ring of Laurent polynomials (in  $n$  or  $2n$  variables), Fock and Goncharov calls  $\mathbb{L}_{\mathcal{C}}$  the ring of ‘universally Laurent polynomials’;  $\mathbb{L}$  stands for the ‘L’aurant. In fact, they defined such a ring only in the quantum case, as we shall encounter later. In the case of a genuine cluster algebra setting, e.g. for  $\mathcal{A}$ -seeds,  $\mathbb{L}_{\mathcal{C}}$  is called the upper cluster algebra.

Fock and Goncharov say that thus finally obtained variety  $\text{Spec}(\mathbb{L}_{\mathcal{C}})$  is the ‘affine closure’ of the one constructed just by gluing the tori. Although taking an affine closure may guarantee somewhat nicer topological properties, I am not sure whether it preserves the scheme structures, which I would like to keep for the sake of upcoming quantization, in the cases of  $\mathcal{X}$ -variety and  $\mathcal{D}$ -variety. So, in the present article, I stick to Def.23 without taking the ‘affine closure’.

**Remark 26.** If there is only one torus for one feed, which possibly underlies many seeds, then this feed torus may be glued to itself in weird ways, creating potential analytical or even topological problems. So we better associate a torus to a seed.

**Remark 27.** The ‘Laurent phenomenon’ of cluster algebras [FZ02] says that when  $\Gamma$  is an  $\mathcal{A}$ -seed, then for each  $\mathcal{A}$ -seed  $\Gamma'$  equivalent to  $\Gamma$ , the image under  $\mu_{\Gamma, \Gamma'}^*$  of each cluster  $\mathcal{A}$ -variable of  $\Gamma'$  is in  $\mathbb{L}_{\Gamma}$ . The subalgebra of  $\mathbb{L}_{\Gamma}$  generated by all these images is the cluster algebra for the seed  $\Gamma$ . So, there are ‘many’ global regular functions the cluster  $\mathcal{A}$ -variety. If we had declared that the ring of regular functions on the local patch associated to a seed should be the ring of polynomials for example, instead of Laurent polynomials, then there would not be so many globally regular functions.

The cluster varieties defined in Def.23 can be formulated in the language of functors on the (saturated) cluster modular groupoids considered in §2.3. Little more generally, a scheme with an atlas whose charts (or, the corresponding ‘local patches’) are enumerated by  $\mathcal{C}$  can be thought of as a contravariant functor

$$\eta : \text{the cluster modular groupoid } \mathcal{G}_{\mathcal{C}} \longrightarrow \text{certain category of commutative rings},$$

where  $\mathcal{G}_{\mathcal{C}}$  stands for one of  $\mathcal{G}_{\mathcal{C}}^{\mathcal{A}}$ ,  $\mathcal{G}_{\mathcal{C}}^{\mathcal{X}}$ , and  $\mathcal{G}_{\mathcal{C}}^{\mathcal{D}}$ , while morphisms from an object to another in the category in the RHS are homomorphisms from the field of fraction of the latter object to that of the former. For our situation, one may further require that the objects of the RHS category must be isomorphic to the ring of Laurent polynomials in  $n$  (or  $2n$ ) variables over  $\mathbb{Z}$ , and that we only allow homomorphisms whose image on the generators can be written

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This may not be a standard usage of the term ‘affine closure’.

as rational expressions using only additions, multiplications, and divisions (i.e. ‘subtraction-free’). Then the local patch corresponding to an element of  $\mathcal{C}$  is set to be the Spec of the image of that element under  $\eta$ , and we glue the patches corresponding to each two elements of  $\mathcal{C}$  by the birational map given by the image under  $\eta$  of the unique morphism in  $\mathcal{G}_{\mathcal{C}}$  from the first to the latter; recall from Lem.20 that in  $\mathcal{G}_{\mathcal{C}}$  there is exactly one morphism from any object to an object. Such functor  $\eta$  is a slight reformulation of Fock-Goncharov’s ‘positive space’ [FG09b, Def.1.3], in which the objects in the target category (denoted by Pos) are split algebraic tori and the morphisms are positive rational maps between the tori.

This groupoid formulation of the cluster varieties is not just an unnecessary luxury to have, for such a formulation in terms of groupoids and rings is the only known way to describe the quantum versions of cluster varieties; there is no actual topological space in the quantum world.

The following geometric structures, which are extra data on the cluster varieties, are crucial in the story of quantization.

**Lemma 28** (geometric structures on cluster varieties). *The geometric structures on the seed tori defined in Def.5 induce well-defined corresponding geometric structures on the respective cluster varieties.*

One checks this lemma by verifying that  $\mu_k$  and  $P_\sigma$  preserve the geometric structures on the seed tori; I omit the computation.

### 3. ALGEBRAIC QUANTIZATION

In [FG09] and their previous works, Fock and Goncharov obtained certain ‘deformation quantization’ of  $\mathcal{X}$ -variety ‘along’ its Poisson structure and that of  $\mathcal{D}$ -variety ‘along’ its symplectic structure; some version of the former follows from the latter, which I think is more canonical. Instead of giving a comprehensive explanation of the meaning of ‘deformation quantization’ in general, I only present here what were obtained in [FG09].

**3.1. Basic formulas for algebraic quantization.** To a  $\mathcal{D}$ -seed  $\Gamma$  we associate the following family of non-commutative associative  $*$ -algebras that deforms the coordinate ring of the seed  $\mathcal{D}$ -torus  $\mathcal{D}_\Gamma$  ‘in the direction of’ its symplectic or Poisson structure.

**Definition 29** (quantum torus algebra for a seed  $\mathcal{D}$ -torus). *Let  $\Gamma = (\varepsilon, d, \{B_i, X_i\}_{i=1}^n)$  be a  $\mathcal{D}$ -seed, and  $N$  be the smallest positive integer such that  $\widehat{\varepsilon}_{ij} = \varepsilon_{ij}/d_j \in \frac{1}{N}\mathbb{Z}, \forall i, j$ . The seed quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_\Gamma^q$  associated to  $\Gamma$  for a quantum parameter  $q$ , which can be regarded as a formal parameter at the moment, is the free associative  $*$ -algebra over  $\mathbb{Z}[q^{1/N}, q^{-1/N}]$  generated by  $\mathbf{B}_1, \dots, \mathbf{B}_n, \mathbf{X}_1, \dots, \mathbf{X}_n$  and their inverses, mod out by the relations*

$$\begin{aligned} q_j^{-\varepsilon_{ij}} \mathbf{X}_i \mathbf{X}_j &= q_i^{-\varepsilon_{ji}} \mathbf{X}_j \mathbf{X}_i, & \forall i, j \in \{1, \dots, n\}, \\ q_i^{-1} \mathbf{X}_i \mathbf{B}_i &= q_i \mathbf{B}_i \mathbf{X}_i, & \forall i \in \{1, \dots, n\}, \\ \mathbf{B}_i \mathbf{X}_j &= \mathbf{X}_j \mathbf{B}_i, & \text{whenever } i \neq j, \\ \mathbf{B}_i \mathbf{B}_j &= \mathbf{B}_j \mathbf{B}_i, & \forall i, j \in \{1, \dots, n\}, \end{aligned}$$

where

$$(3.1) \quad q_i := (q^{1/N})^{N/d_i}, \quad \forall i \in \{1, \dots, n\},$$

with the  $*$ -structure defined as the unique ring anti-homomorphism satisfying

$$*\mathbf{X}_i = \mathbf{X}_i, \quad *\mathbf{B}_i = \mathbf{B}_i, \quad \forall i \in \{1, \dots, n\}, \quad *q^{1/N} = q^{-1/N}.$$

Denote by  $\mathbb{D}_\Gamma^q$  the skew field of fractions of  $\mathbf{D}_\Gamma^q$ .

**Remark 30.** Eq.(3.1) is a careful way of saying  $q_i = q^{1/d_i}$ .

I think that it is a mistake that the relations  $\mathbf{B}_i \mathbf{B}_j = \mathbf{B}_j \mathbf{B}_i$ , which do not follow from others, are omitted in eq.(58) in p.252 [FG09, §3].

**Definition 31** (tilde variables for the seed quantum  $\mathcal{D}$ -torus algebra). *For each  $i \in \{1, \dots, n\}$ , define an element  $\widetilde{\mathbf{X}}_i$  of  $\mathbf{D}_\Gamma^q$  by*

$$\widetilde{\mathbf{X}}_i := \mathbf{X}_i \prod_{j=1}^n \mathbf{B}_j^{\varepsilon_{ij}}.$$

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One could just use  $N =$  the least common multiple of all  $d_i$ ’s.

I use bold faced letters in order to distinguish from the ‘classical’ cluster variables.

‘skew field’ is a synonym of ‘division ring’

Then one can check

$$q_j^{\varepsilon_{ij}} \tilde{\mathbf{X}}_i \tilde{\mathbf{X}}_j = q_i^{\varepsilon_{ji}} \tilde{\mathbf{X}}_j \tilde{\mathbf{X}}_i, \quad q_i^{-\delta_{i,j}} \tilde{\mathbf{X}}_i \mathbf{B}_j = q_i^{\delta_{i,j}} \mathbf{B}_j \tilde{\mathbf{X}}_i, \quad \mathbf{X}_i \tilde{\mathbf{X}}_j = \tilde{\mathbf{X}}_j \mathbf{X}_i,$$

for all  $i, j \in \{1, \dots, n\}$ .

To a mutation  $\Gamma \xrightarrow{k} \Gamma'$  we associate the following  $*$ -isomorphism  $\mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_{\Gamma}^q$  of skew fields.

**Definition 32** (quantum mutation map). *Let the two  $\mathcal{D}$ -seeds  $\Gamma = (\varepsilon, d, *)$  and  $\Gamma' = (\varepsilon', d', *)$  be related by the mutation along  $k$ , that is,  $\Gamma' = \mu_k(\Gamma)$ . Define the map  $\mu_k^q : \mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_{\Gamma}^q$  by*

$$\mu_k^q := \mu_k^{\sharp q} \circ \mu'_k,$$

where  $\mu_k^{\sharp q}$  is the automorphism of  $\mathbb{D}_{\Gamma}^q$  given by the following formulas on generators

$$(3.2) \quad \mu_k^{\sharp q}(\mathbf{B}_i) = \begin{cases} \mathbf{B}_i & \text{if } i \neq k, \\ \mathbf{B}_k(1 + q_k \mathbf{X}_k)(1 + q_k \tilde{\mathbf{X}}_k)^{-1} & \text{if } i = k, \end{cases}$$

$$(3.3) \quad \mu_k^{\sharp q}(\mathbf{X}_i) = \mathbf{X}_i \prod_{r=1}^{|\varepsilon_{ik}|} (1 + (q_k^{\text{sgn}(-\varepsilon_{ik})})^{2r-1} \mathbf{X}_k)^{\text{sgn}(-\varepsilon_{ik})}, \quad \forall i \in \{1, \dots, n\},$$

and  $\mu'_k$  is induced by the map  $\mu'_k : \mathbf{D}_{\Gamma'}^q \rightarrow \mathbf{D}_{\Gamma}^q$  given by

$$\mu'_k(\mathbf{B}'_i) = \begin{cases} \mathbf{B}_i & \text{if } i \neq k, \\ \mathbf{B}_k^{-1} \prod_{j=1}^n \mathbf{B}_j^{[-\varepsilon_{kj}]_+} & \text{if } i = k, \end{cases} \quad \mu'_k(\mathbf{X}'_i) = \begin{cases} q_k^{-\varepsilon_{ik}[\varepsilon_{ik}]_+} \mathbf{X}_i (\mathbf{X}_k)^{[\varepsilon_{ik}]_+} & \text{if } i \neq k, \\ \mathbf{X}_k^{-1} & \text{if } i = k, \end{cases}$$

on the generators of  $\mathbf{D}_{\Gamma'}^q$ , where  $[\ ]_+$  denotes the ‘positive part’ of a real number:

$$[a]_+ := \frac{a + |a|}{2} = \begin{cases} a, & \text{if } a \geq 0, \\ 0, & \text{otherwise} \end{cases}, \quad \forall a \in \mathbb{R}.$$

**Definition 33** (quantum permutation map). *Let  $\Gamma' = P_{\sigma}(\Gamma)$  for a permutation  $\sigma$  of  $\{1, \dots, n\}$ . Define the map  $\mathbb{P}_{\sigma} : \mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_{\Gamma}^q$  to be the one induced by the map  $\mathbb{P}_{\sigma} : \mathbf{D}_{\Gamma'}^q \rightarrow \mathbf{D}_{\Gamma}^q$  given by*

$$\mathbb{P}_{\sigma}(\mathbf{B}'_{\sigma(i)}) = \mathbf{B}_i, \quad \mathbb{P}_{\sigma}(\mathbf{X}'_{\sigma(i)}) = \mathbf{X}_i, \quad \forall i \in \{1, \dots, n\}.$$

For a quantization, we would like to assign to each cluster transformation a well-defined quantum map between the skew fields of fractions of the respective seed quantum  $\mathcal{D}$ -torus algebras, such that 1) it is a  $*$ -isomorphism of skew fields, and 2) it recovers the classical map between the seed  $\mathcal{D}$ -tori in the classical limit  $q \rightarrow 1$ . The above defined maps  $\mu_k^q$  and  $\mathbb{P}_{\sigma}$  are what we assign to elementary cluster transformations, and it is a straightforward task to check that these satisfy the two conditions 1) and 2) just mentioned.

A general cluster transformation is the composition of a sequence of elementary ones, and we would assign to it the composition of the sequence of corresponding quantum maps  $\mu_k^q$  and  $\mathbb{P}_{\sigma}$  for the elementary cluster transformations, although in a reverse order, for the maps  $\mu_k^q$  and  $\mathbb{P}_{\sigma}$  are constructed in a ‘contravariant’ manner. Problem is, there can be several sequences of elementary cluster transformations expressing the same classical cluster transformation, and we have to make sure that these sequences yield the same quantum map. Equivalently, we must make sure that each trivial cluster transformation is assigned the identity map of the relevant quantum  $\mathcal{D}$ -torus algebra. This ‘consistency’ is the key aspect in the algebraic part of the program of quantization of cluster varieties. A standard way of formulating this problem is via the (saturated) cluster modular groupoids considered in §2.3, and we do this in the following subsection.

### 3.2. Groupoid formulation of algebraic quantization.

**Definition 34.** *The category of seed quantum  $\mathcal{D}$ -torus algebras  $\text{QDTor}^q = \text{QDTor}_{\mathcal{C}}^q$  associated to an equivalence class  $\mathcal{C}$  of  $\mathcal{D}$ -seeds with quantum parameter  $q$  is a category whose objects are  $\mathbf{D}_{\Gamma}^q$  with  $\Gamma$  in the equivalence class  $\mathcal{C}$ , and the set morphisms  $\text{Hom}_{\text{QDTor}^q}(\mathbf{D}_{\Gamma}^q, \mathbf{D}_{\Gamma'}^q)$  is the set of all  $*$ -isomorphisms from the skew field  $\mathbb{D}_{\Gamma'}^q$  to the skew field  $\mathbb{D}_{\Gamma}^q$  whose images of the generators of  $\mathbb{D}_{\Gamma'}^q$  are subtraction-free elements of  $\mathbb{D}_{\Gamma}^q$ . (see Def.29 for  $\mathbf{D}_{\Gamma}^q$  and  $\mathbb{D}_{\Gamma}^q$ .)*

Here, a ‘subtraction-free’ element is one that can be expressed in terms of generators and  $q^{\pm 1/N}$  using only additions, multiplications, and divisions.

**Definition 35.** By a quantum  $\mathcal{D}$ -space  $\mathcal{D}^q = \mathcal{D}_{\mathcal{C}}^q$ , or quantum double, for a cluster  $\mathcal{D}$ -variety  $\mathcal{D} = \mathcal{D}_{\mathcal{C}}$  with quantum parameter  $q$  we mean a contravariant functor

$$(3.4) \quad \eta^q : \text{the saturated cluster modular groupoid } \widehat{\mathcal{G}}_{\mathcal{C}}^{\mathcal{D}} \longrightarrow \text{QDTor}_{\mathcal{C}}^q$$

whose images of elementary morphisms recover the formulas in Definitions 8 and 10 in the ‘classical limit’  $q \rightarrow 1$ .

Let us call such a functor  $\eta^q$  an algebraic quantum cluster ( $\mathcal{D}$ -)variety.

**Remark 36.** [FG09] views a functor  $\eta^q$  in (3.4) as a ‘quantum scheme’.

Of course, an ultimate goal to attain is a functor (3.4) from the cluster modular groupoid  $\mathcal{G}_{\mathcal{C}}^{\mathcal{D}}$ , not just from the saturated cluster modular groupoid  $\widehat{\mathcal{G}}_{\mathcal{C}}^{\mathcal{D}}$ . What has been obtained so far is only (3.4). We know the images of the objects (Def.29), as well as images of the elementary morphisms (Def.32, 33). Hence we also know the images of general morphisms, for they are sequences of elementary ones. The only problem is then the well-definedness of this construction.

**Lemma 37** (Lem.3.4 of [FG09]). *The functor  $\eta^q$  described above is well-defined.*

Fock-Goncharov [FG09] says that this can be obtained either by direct computation, or by using the results of [BZ05, Thm.6.1]. (In the present paper we shall prove a different statement that imply these relations.)

#### 4. REVIEW ON FUNCTIONAL ANALYSIS

As is often the case in a ‘physical’ quantization, one does not stop at the non-commutative algebras, but considers representations of them on Hilbert spaces, respecting the  $*$ -structure. In the present section we review some definitions and lemmas from functional analysis, without proofs. I refer the readers to [RS80] and [Y80], which are great sources to learn the subject, and from which I collected most of the definitions and statements for this section. In [K16] one can find a more detailed treatment with some proofs.

All Hilbert spaces here are over  $\mathbb{C}$ . For a Hilbert space  $V$  we denote by

$$\langle \cdot, \cdot \rangle_V : V \times V \rightarrow \mathbb{C}$$

its inner product, which is complex linear in the first argument and conjugate linear in the second argument. The norm

$$\|v\|_V := \sqrt{\langle v, v \rangle_V}, \quad \forall v \in V$$

on  $V$  defines a topology on  $V$ , which we refer to as the *standard Hilbert space topology* on  $V$ .

**Definition 38** (densely defined operators on a Hilbert space). *Let  $V$  be a Hilbert space, and let  $T$  be a  $\mathbb{C}$ -linear map  $T : D(T) \rightarrow V$  for a  $\mathbb{C}$ -vector subspace  $D(T)$  of  $V$  that is dense in  $V$  with respect to the standard Hilbert space topology on  $V$ . We say that  $D(T)$  is the domain of  $T$  and that  $T$  is a densely defined operator on  $V$ .*

**Remark 39.** *The domain is also a part of the data of a ‘densely defined operator’.*

**Definition 40.** *Let  $D$  be a  $\mathbb{C}$ -linear subspace of a Hilbert space  $V$ , and let  $T : D \rightarrow V$  be a  $\mathbb{C}$ -linear map. We say that  $D$  is invariant under  $T$  or  $T$  leaves  $D$  invariant if  $Tv \in D$ ,  $\forall v \in D$ .*

**Definition 41.** *Let  $T_1, T_2$  be densely defined operators on a Hilbert space  $V$ . We say that  $T_2$  extends  $T_1$  (or  $T_2$  is an extension of  $T_1$ ) if  $D(T_1) \subset D(T_2)$  and  $T_1v = T_2v$  for all  $v \in D(T_1)$ . We write this as*

$$T_1 \subset T_2.$$

**Definition 42.** *A densely defined operator  $T$  on a Hilbert space  $V$  is said to be symmetric if*

$$\langle Tv, w \rangle_V = \langle v, Tw \rangle_V, \quad \forall v, w \in D(T).$$

Let us now review the definition of the *adjoint* of a densely defined operator. Let  $T$  be a densely defined operator on a Hilbert space  $V$ , with its domain  $D(T)$ . Consider  $v \in V$  such that the following holds:

$$(4.1) \quad w \mapsto \langle Tw, v \rangle_V \text{ is a bounded linear functional on } D(T).$$

Then the Riesz Representation Theorem (or the ‘Riesz Lemma’) tells us that there exists a unique vector  $u$  in  $V$  such that  $\langle Tw, v \rangle_V = \langle w, u \rangle_V$  for all  $w \in D(T)$ , for  $D(T)$  is dense in  $V$ . Denote such  $u$  by  $u = T^*v$ . Thus we obtain a certain  $\mathbb{C}$ -linear map  $T^* : D(T^*) \rightarrow V$  where

$$D(T^*) := \{v \in V \mid (4.1) \text{ holds}\},$$

which is not necessarily dense in  $V$ .

**Definition 43** (adjoint of a densely defined operator). *Let  $T$  be a densely defined operator on a Hilbert space  $V$ . The unique  $\mathbb{C}$ -linear map  $D(T^*) \rightarrow V$  constructed above is called the adjoint of  $T$ .*

Obviously we have  $T \subset T^*$  for each symmetric operator  $T$ . In fact this is a characterization of symmetric operators among densely defined operators.

**Definition 44** (self-adjoint and essentially self-adjoint operators). *We say that a densely operator  $T$  on a Hilbert space  $V$  is self-adjoint if and only if  $T = T^*$ .*

*A densely defined symmetric operator  $T$  is called essentially self-adjoint if and only if it has a unique self-adjoint extension.*

Densely defined self-adjoint operators on Hilbert spaces are one of the most basic objects of study in quantum theory. However, as it is tricky to deal with their domains and ranges, we often turn the situation into the story of another class of operators, which are far more nicely behaved.

**Definition 45.** A unitary operator on a Hilbert space  $V$  is a bijective  $\mathbb{C}$ -linear operator  $T : V \rightarrow V$  such that  $\langle Tv, Tw \rangle_V = \langle v, w \rangle_V, \forall v, w \in V$ .

In particular, a unitary operator on  $V$  is defined on the whole  $V$ . If  $T$  is a densely defined operator on  $V$  satisfying  $\langle Tv, Tw \rangle_V = \langle v, w \rangle_V$  for all  $v, w$  in its domain, we can uniquely extend  $T$  to the whole  $V$ , for it is continuous, i.e. bounded (use the B.L.T. theorem; see e.g. [RS80, Thm.I.7]); we will then identify such  $T$  with its unique extension.

What are called the *spectral theorem* and the *functional calculus* allow us to define a well-defined densely defined operator  $f(T)$  for each densely-defined self-adjoint operator  $T$  and a complex-valued continuous function  $f$  on  $\mathbb{R}$  (in a more general version,  $f$  is allowed to be merely a measurable function on some subset of  $\mathbb{R}$ ); in particular, the domain  $D(f(T))$  is also determined by  $T$  and  $f$ . In [K16l] I collected definitions and statements necessary to understand this, which I omit here. One example is that, if  $T$  is given by multiplication by a function, say  $F$ , then  $f(T)$  is given by multiplication by  $f(F) = f \circ F$  on a suitable dense domain. A rough heuristic idea is that on each ‘eigenspace’ of  $T$  for ‘eigenvalue  $\lambda$ ’, the operator  $f(T)$  is defined in such a way that it acts as  $f(\lambda)$  times identity.

**Lemma 46** (unitary operators from functional calculus). *A complex-valued function on a measure space or a topological space is called a unitary function if its values lie in  $U(1) \subset \mathbb{C}$ , i.e. are of modulus 1. Suppose that  $f$  is a unitary function on  $\mathbb{R}$  and  $T$  is a (densely-defined) self-adjoint operator on a Hilbert space  $V$ . Then the operator  $f(T)$ , defined by the functional calculus, is unitary.* ■

For a densely defined self-adjoint operator  $T$ , there are two standard ways to obtain unitary operators. One is to consider the family  $e^{\sqrt{-1}tT}$ ,  $t \in \mathbb{R}$ , of unitary operators (Stone’s Theorem [S32]), and the other is to consider the ‘Cayley transform’  $(T - \sqrt{-1} \cdot \text{id})(T + \sqrt{-1} \cdot \text{id})^{-1}$ , which is also a unitary operator. For more detailed treatments, see [RS80] and [Y80], or [K16l, §4].

We end this section with a useful lemma, to be used later.

**Lemma 47** (unitary conjugation commutes with functional calculus). *If  $T$  is a densely defined self-adjoint operator on a Hilbert space  $V$  with its domain  $D(T)$ , and if  $U$  is a unitary operator on  $V$ , then  $UTU^{-1}$  is a densely defined self-adjoint operator on  $V$  with domain  $D(UTU^{-1}) = U(D(T)) := \{Uv \mid v \in D(T)\}$ . For any complex-valued continuous function  $f$  on  $\mathbb{R}$ , the densely defined operators  $f(UTU^{-1})$  and  $f(T)$ , obtained by the functional calculus applied to  $UTU^{-1}$  and  $T$  respectively with the same function  $f$ , are related by*

$$f(UTU^{-1}) = Uf(T)U^{-1},$$

where  $D(f(UTU^{-1})) = U(D(f(T)))$ . ■

## 5. REPRESENTATIONS OF ALGEBRAIC QUANTUM CLUSTER VARIETIES

**5.1. Positive integrable representations of quantum torus algebras.** For a representation of a seed quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_\Gamma^q$ , we would like to look for operators  $\widehat{\mathbf{B}}_i = \pi_\Gamma(\mathbf{B}_i)$ ,  $\widehat{\mathbf{X}}_i = \pi_\Gamma(\mathbf{X}_i)$ ,  $i = 1, \dots, n$ , on a Hilbert space  $\mathcal{H}_\Gamma$ , that satisfy the algebraic relations of  $\mathbf{B}_i$  and  $\mathbf{X}_i$ . Since  $\mathbf{B}_i$ ,  $\mathbf{X}_i$ , and their corresponding operators, are supposed to be quantum counterparts of the positive real valued functions  $B_i$ ,  $X_i$  on the space  $\mathcal{D}_\mathcal{C}(\mathbb{R}_{>0})$ , i.e.

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The notion of self-adjoint operators applies only to densely defined operators, so we might just say ‘self-adjoint operators’ for densely defined self-adjoint operators.

not a standard terminology

the positive real points of our cluster  $\mathcal{D}$ -variety, it is natural to require that the operators be (densely defined) self-adjoint operators that are positive-definite.

As in usual quantization stories,  $\mathcal{H}_\Gamma$  is the space of all square-integrable functions on some Euclidean (measure) space  $\mathbb{R}^n$ , and each of these positive-definite self-adjoint operators are defined only on a dense subspace of  $\mathcal{H}_\Gamma$ . So it is a tricky matter to deal with the commutation relations of such operators, for we have to be careful about the domains and ranges of these operators. I suggest the following definition as a way to define and characterize certain ‘nicely-behaved’ class of representations of the quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_\Gamma^q$ , using the ‘Weyl relations’, as employed already by von Neumann.

**Definition 48.** Let  $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ , and let

$$q = e^{\pi\sqrt{-1}\hbar}.$$

A positive integrable  $*$ -representation of the seed quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_\Gamma^q$  is a triple  $(\mathcal{H}_\Gamma, D_\Gamma, \pi_\Gamma^q)$ , where  $\mathcal{H}_\Gamma$  is a Hilbert space,  $D_\Gamma$  is a dense  $\mathbb{C}$ -linear subspace of  $\mathcal{H}_\Gamma$ , and  $\pi_\Gamma^q$  consists of essentially self-adjoint operators  $\widehat{\mathbf{b}}_i^\hbar$ ,  $\widehat{\mathbf{x}}_i^\hbar$ ,  $i = 1, \dots, n$ , on  $D_\Gamma$ , such that

1) for any

$$(5.1) \quad (\alpha) = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \quad \text{and} \quad (\beta) = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$$

the following unitary operators on  $\mathcal{H}_\Gamma$

$$\widehat{\mathbf{B}}_i^{(\alpha)} := e^{\sqrt{-1}\alpha_i \widehat{\mathbf{b}}_i^\hbar}, \quad \widehat{\mathbf{X}}_i^{(\beta)} := e^{\sqrt{-1}\beta_i \widehat{\mathbf{x}}_i^\hbar}, \quad i = 1, \dots, n,$$

defined by the functional calculus for the unique self-adjoint extensions of the operators  $\widehat{\mathbf{b}}_i^\hbar$ ’s and  $\widehat{\mathbf{x}}_i^\hbar$ ’s with the (unitary) functions  $\lambda \mapsto e^{\sqrt{-1}\alpha_i \lambda}$  and  $\lambda \mapsto e^{\sqrt{-1}\beta_i \lambda}$  respectively, leave  $D_\Gamma$  invariant and satisfy the relations

$$(5.2) \quad \begin{cases} e^{\pi\sqrt{-1}\hbar_j \varepsilon_{ij} \beta_i \beta_j} \widehat{\mathbf{X}}_i^{(\beta)} \widehat{\mathbf{X}}_j^{(\beta)} &= e^{\pi\sqrt{-1}\hbar_i \varepsilon_{ji} \beta_j \beta_i} \widehat{\mathbf{X}}_j^{(\beta)} \widehat{\mathbf{X}}_i^{(\beta)}, & \forall i, j \in \{1, \dots, n\}, \\ e^{\pi\sqrt{-1}\hbar_i \beta_i \alpha_i} \widehat{\mathbf{X}}_i^{(\beta)} \widehat{\mathbf{B}}_i^{(\alpha)} &= e^{-\pi\sqrt{-1}\hbar_i \alpha_i \beta_i} \widehat{\mathbf{B}}_i^{(\alpha)} \widehat{\mathbf{X}}_i^{(\beta)}, & \forall i \in \{1, \dots, n\}, \\ \widehat{\mathbf{B}}_i^{(\alpha)} \widehat{\mathbf{X}}_j^{(\beta)} &= \widehat{\mathbf{X}}_j^{(\beta)} \widehat{\mathbf{B}}_i^{(\alpha)}, & \text{whenever } i \neq j, \\ \widehat{\mathbf{B}}_i^{(\alpha)} \widehat{\mathbf{B}}_j^{(\alpha)} &= \widehat{\mathbf{B}}_j^{(\alpha)} \widehat{\mathbf{B}}_i^{(\alpha)}, & \forall i, j \in \{1, \dots, n\}, \end{cases}$$

where

$$(5.3) \quad \hbar_i := \hbar/d_i, \quad \forall i = 1, \dots, n, \quad \text{and}$$

2) the following densely defined self-adjoint operators on  $\mathcal{H}_\Gamma$

$$(5.4) \quad \pi_\Gamma^q(\mathbf{B}_i^{\pm 1}) := e^{\pm \widehat{\mathbf{b}}_i^\hbar}, \quad \pi_\Gamma^q(\mathbf{X}_i^{\pm 1}) := e^{\pm \widehat{\mathbf{x}}_i^\hbar}, \quad i = 1, \dots, n,$$

defined by the functional calculus for the unique self-adjoint extensions of the operators  $\widehat{\mathbf{b}}_i^\hbar$ ’s and  $\widehat{\mathbf{x}}_i^\hbar$ ’s with the functions  $e^{\pm \lambda}$  respectively, leave  $D_\Gamma$  invariant.

Few words on the terminology ‘positive integrable  $*$ -representation’. The word ‘integrable’ hints to the ‘Weyl relations’ (5.2). In this regard, note that the ‘Weyl relations’ (5.2) imply the ‘Heisenberg relations’

$$(5.5) \quad [\widehat{\mathbf{x}}_i^\hbar, \widehat{\mathbf{x}}_j^\hbar] = 2\pi\sqrt{-1}\hbar_j \varepsilon_{ij} \cdot \text{id}, \quad [\widehat{\mathbf{x}}_i^\hbar, \widehat{\mathbf{b}}_j^\hbar] = 2\pi\sqrt{-1}\hbar_i \delta_{i,j} \cdot \text{id}, \quad [\widehat{\mathbf{b}}_i^\hbar, \widehat{\mathbf{b}}_j^\hbar] = 0, \quad \forall i, j,$$

on  $D_\Gamma$  (see e.g. Corollary of Theorem VIII.14 of [RS80]), where  $\delta_{i,j}$  is the Kronecker delta, but the ‘Heisenberg relations’ (5.5) may not imply the ‘Weyl relations’ (5.2) in a general Hilbert space; see ‘Nelson’s example’ and its perturbation in [RS80, Chap.VIII.5]. Finally, one can prove that the operators (5.4) satisfy the algebraic relations satisfied by the corresponding generators of  $\mathbf{D}_\Gamma^q$ , when applied to elements in  $D_\Gamma$ . One way of showing this is to use the ‘Stone-von Neumann Theorem’ ([v31]; see e.g. Thm.VIII.14 of [RS80]) for each  $i = 1, \dots, n$  (one at a time), to write down  $\widehat{\mathbf{b}}_i$  and  $\widehat{\mathbf{x}}_i$  as explicit operators on an explicit Hilbert space, and check directly; for instance, one can check for the concrete examples of positive integrable  $*$ -representations that I will soon present. So,  $\pi_\Gamma$  can indeed be thought of as a ‘representation’ of the algebra  $\mathbf{D}_\Gamma^q$ . Notice that Fock and Goncharov use just (5.5) in [FG09], instead of the full Weyl relations.

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the superscripts  $\hbar$  in  $\widehat{\mathbf{b}}_i^\hbar$  and  $\widehat{\mathbf{x}}_i^\hbar$  do not mean powers, but just indicate the associated parameter  $\hbar$

The ‘tilde’ operators are defined by the formula

$$(5.6) \quad \widehat{\mathbf{x}}_i^{\hbar} := \widehat{\mathbf{x}}_i^{\hbar} + \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{b}}_j^{\hbar}, \quad \pi_{\Gamma}^q(\widetilde{\mathbf{X}}_i^{\pm 1}) := e^{\pm \widehat{\mathbf{x}}_i^{\hbar}}, \quad \forall i = 1, \dots, n,$$

as essentially self-adjoint operators on  $D_{\Gamma}$ .

Def.48 is designed in such a way to deal with the analytical issues for the algebraic relations that must be satisfied by the positive-definite self-adjoint operators corresponding to the generators of  $\mathbf{D}_{\Gamma}^q$ . At the moment, let us not bother about the operators corresponding to other elements of the skew field of fractions  $\mathbb{D}_{\Gamma}^q$  of  $\mathbf{D}_{\Gamma}^q$ ; we will later discuss how Fock and Goncharov [FG09] dealt with the analytical issues for the operators for sufficiently many elements of  $\mathbb{D}_{\Gamma}^q$  simultaneously.

**Definition 49.** A positive integrable  $*$ -representation  $(\mathcal{H}_{\Gamma}, D_{\Gamma}, \pi_{\Gamma}^q)$  of the seed quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_{\Gamma}^q$  is said to be weakly irreducible if  $D_{\Gamma}$  has no nonzero  $\mathbb{C}$ -vector subspace  $D$  that is invariant under all  $\widehat{\mathbf{B}}_i^{(\alpha)}$ ,  $\widehat{\mathbf{X}}_i^{(\beta)}$ ,  $\pi_{\Gamma}^q(\mathbf{B}_i^{\pm 1})$ ,  $\pi_{\Gamma}^q(\mathbf{X}_i^{\pm 1})$ ,  $i = 1, \dots, n$ , such that the closure of  $D$  in  $\mathcal{H}_{\Gamma}$  is a proper Hilbert subspace of  $\mathcal{H}_{\Gamma}$ .

Let us now present two examples of weakly irreducible positive integrable  $*$ -representations of  $\mathbf{D}_{\Gamma}^q$ , due to Fock-Goncharov [FG09], with a slight modification.

**Definition 50.** For a  $\mathcal{D}$ -seed  $\Gamma$ , define the Hilbert space

$$(5.7) \quad \mathcal{H}_{\Gamma} := L^2(\mathbb{R}^n, da_1 da_2 \dots da_n),$$

where the measure on  $\mathbb{R}^n$  is the product of Lebesgue measures  $da_i$  on  $\mathbb{R}$ , and the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\Gamma}}$  is the usual one

$$\langle f, g \rangle_{\mathcal{H}_{\Gamma}} = \int_{\mathbb{R}^n} f \bar{g} da_1 \dots da_n.$$

Define  $D_{\Gamma}$  as the following  $\mathbb{C}$ -vector subspace of  $\mathcal{H}_{\Gamma}$ :

$$(5.8) \quad D_{\Gamma} := \text{span}_{\mathbb{C}} \left\{ e^{-\sum_{i=1}^n (\alpha_i a_i^2 + \beta_i a_i)} P(a_1, \dots, a_n) \mid \begin{array}{l} \alpha_i, \beta_i \in \mathbb{C}, \text{ Re}(\alpha_i) > 0, \text{ and } P \text{ is} \\ \text{a polynomial in } a_1, \dots, a_n \text{ over } \mathbb{C} \end{array} \right\}.$$

**Remark 51.** The definitions of  $\mathcal{H}_{\Gamma}$  and  $D_{\Gamma}$  do not really use the data of a  $\mathcal{D}$ -seed  $\Gamma$ , except for the number  $n$ . The subscript  $\Gamma$  is mainly for the sake of keeping track of notations. Readers will find this useful later when we deal with multiple  $\Gamma$ 's.

Recall that an element of  $\mathcal{H}_{\Gamma}$  is the equivalence class of a square-integrable measurable function on  $\mathbb{R}^n$ , where two functions are set to be equivalent if they coincide except at a set of measure zero. Usually we will be working with one representative for each equivalence class.

**Lemma 52.** In Def.50,  $D_{\Gamma}$  is dense in  $\mathcal{H}_{\Gamma}$ , with respect to the standard Hilbert space topology.  $\blacksquare$

This  $D_{\Gamma}$  contains Fock-Goncharov's dense subspace in [FG09], which is denoted by  $W_1$  there; they require  $\alpha_1 = \alpha_2 = \dots = \alpha_n > 0$ . One nice property of  $D_{\Gamma}$  (or  $W_1$ ) is:

**Lemma 53.** Let  $\mathcal{F}_i : \mathcal{H}_{\Gamma} \rightarrow \mathcal{H}_{\Gamma}$  be the Fourier transform in the  $a_i$  variable; more precisely, on  $L^1(\mathbb{R}^n, da_1 \dots da_n) \cap \mathcal{H}_{\Gamma}$  it is given by the formula

$$(5.9) \quad (\mathcal{F}_i f)(a_1, \dots, a_i, \dots, a_n) = \int_{\mathbb{R}} e^{-2\pi\sqrt{-1}a_i b_i} f(a_1, \dots, b_i, \dots, a_n) db_i.$$

Then,  $\mathcal{F}_i(D_{\Gamma}) = D_{\Gamma}$ .  $\blacksquare$

**Definition 54** (the Schrödinger representation). Let  $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$  and let  $D_{\Gamma}$ ,  $\mathcal{H}_{\Gamma}$  be as in Def.50. Define the operators  $\widehat{\mathbf{q}}_i^{\hbar}$ ,  $\widehat{\mathbf{p}}_i^{\hbar} : D_{\Gamma} \rightarrow \mathcal{H}_{\Gamma}$ ,  $i = 1, \dots, n$ , by the formulas

$$(5.10) \quad \widehat{\mathbf{q}}_i^{\hbar} := a_i \quad \text{and} \quad \widehat{\mathbf{p}}_i^{\hbar} := 2\pi\sqrt{-1}\hbar \frac{\partial}{\partial a_i} \quad \text{on } D_{\Gamma},$$

for each  $i = 1, \dots, n$ , i.e.

$$(\widehat{\mathbf{q}}_i^{\hbar} f)(a_1, \dots, a_n) = a_i f(a_1, \dots, a_n), \quad (\widehat{\mathbf{p}}_i^{\hbar} f)(a_1, \dots, a_n) = 2\pi\sqrt{-1}\hbar \frac{\partial f}{\partial a_i}(a_1, \dots, a_n)$$

for all  $f(a_1, \dots, a_n) \in D_{\Gamma}$ .

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the superscripts  $\hbar$  in  $\widehat{\mathbf{q}}_i^{\hbar}$  and  $\widehat{\mathbf{p}}_i^{\hbar}$  do not mean powers.

It is straightforward to check:

**Lemma 55.** *Each of  $\widehat{\mathbf{q}}_i^h$ 's and  $\widehat{\mathbf{p}}_i^h$ 's leaves  $D_\Gamma$  invariant, and is essentially self-adjoint on  $D_\Gamma$ , in the Hilbert space  $\mathcal{H}_\Gamma$ . On  $D_\Gamma$  they satisfy the ‘Heisenberg relations’*

$$(5.11) \quad [\widehat{\mathbf{p}}_i^h, \widehat{\mathbf{q}}_j^h] = 2\pi\sqrt{-1}\hbar \delta_{i,j} \cdot \text{id}.$$

Moreover, the corresponding strongly continuous unitary groups  $e^{\sqrt{-1}\alpha_i\widehat{\mathbf{q}}_i^h}$ ,  $e^{\sqrt{-1}\beta_i\widehat{\mathbf{p}}_i^h}$ , defined by the functional calculus for the unique self-adjoint extensions of the operators  $\widehat{\mathbf{q}}_i^h$ 's and  $\widehat{\mathbf{p}}_i^h$ 's with the functions  $e^{\sqrt{-1}\alpha_i\lambda}$  and  $e^{\sqrt{-1}\beta_i\lambda}$  respectively, for any real numbers  $\alpha_i$ 's and  $\beta_i$ 's, satisfy the ‘Weyl relations’

$$e^{\sqrt{-1}\alpha_i\widehat{\mathbf{p}}_i^h} e^{\sqrt{-1}\beta_i\widehat{\mathbf{q}}_i^h} = e^{-2\pi\sqrt{-1}\hbar\alpha_i\beta_i} e^{\sqrt{-1}\beta_i\widehat{\mathbf{q}}_i^h} e^{\sqrt{-1}\alpha_i\widehat{\mathbf{p}}_i^h}.$$

**Lemma 56** (new  $*$ -representation of  $\mathbf{D}_\Gamma^q$ ; [FG09, §5]). *Let  $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ , and  $q = e^{\pi\sqrt{-1}\hbar}$ . The following operators defined on the dense subspace  $D_\Gamma$  of the Hilbert space  $\mathcal{H}_\Gamma$  given by*

$$(5.12) \quad \widehat{\mathbf{b}}_i^h = (\widehat{\mathbf{b}}_i^h)^{\text{new}} := \widehat{\mathbf{q}}_i^h, \quad \widehat{\mathbf{x}}_i^h = (\widehat{\mathbf{x}}_i^h)^{\text{new}} := d_i^{-1} \widehat{\mathbf{p}}_i^h - \sum_{j=1}^n [\varepsilon_{ij}]_+ \widehat{\mathbf{q}}_j^h, \quad \forall i = 1, \dots, n,$$

where  $\widehat{\mathbf{q}}_i^h$ 's and  $\widehat{\mathbf{p}}_i^h$ 's are as in Def.54, define a weakly irreducible positive integrable  $*$ -representation of the seed quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_\Gamma^q$ , denoted by  $(\mathcal{H}_\Gamma, D_\Gamma, \pi_\Gamma^q)$ .

This ‘new’ representation is what is used in [FG09] by Fock and Goncharov; they changed from another choice of representation from their previous paper [FG07]. We will soon see that opting to use this new representation was not a good decision. Anyways, the ‘tilde’ counterpart (5.6) of the new representation is given by

$$\widehat{\mathbf{x}}_i^h = (\widehat{\mathbf{x}}_i^h)^{\text{new}} := (\widehat{\mathbf{x}}_i^h)^{\text{new}} + \sum_{j=1}^n \varepsilon_{ij} (\widehat{\mathbf{b}}_j^h)^{\text{new}} = d_i^{-1} \widehat{\mathbf{p}}_i^h - \sum_{j=1}^n [-\varepsilon_{ij}]_+ \widehat{\mathbf{q}}_j^h \quad (\text{on } D_\Gamma).$$

Their ‘old’ representation is as follows (eq.(78) of [FG09], or [FG07, §4.1]):

**Lemma 57** (old  $*$ -representation of  $\mathbf{D}_\Gamma^q$ ; [FG07]). *Let  $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ , and  $q = e^{\pi\sqrt{-1}\hbar}$ . The following operators defined on the dense subspace  $D_\Gamma$  of the Hilbert space  $\mathcal{H}_\Gamma$  given by*

$$(5.13) \quad (\widehat{\mathbf{b}}_i^h)^{\text{old}} := 2\widehat{\mathbf{q}}_i^h, \quad (\widehat{\mathbf{x}}_i^h)^{\text{old}} := \frac{1}{2}d_i^{-1} \widehat{\mathbf{p}}_i^h - \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{q}}_j^h, \quad \forall i = 1, \dots, n,$$

where  $\widehat{\mathbf{q}}_i^h$ 's are as in (5.3), define a weakly irreducible positive integrable  $*$ -representation of the seed quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_\Gamma^q$ , denoted by  $(\mathcal{H}_\Gamma, D_\Gamma, (\pi_\Gamma^q)^{\text{old}})$ .

Again, the operators for the ‘tilde’ counterpart (5.6) are given by

$$(5.14) \quad (\widehat{\mathbf{x}}_i^h)^{\text{old}} := (\widehat{\mathbf{x}}_i^h)^{\text{old}} + \sum_{j=1}^n \varepsilon_{ij} (\widehat{\mathbf{b}}_j^h)^{\text{old}} = \frac{1}{2}d_i^{-1} \widehat{\mathbf{p}}_i^h + \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{q}}_j^h \quad (\text{on } D_\Gamma).$$

**5.2. The Langlands modular dual.** For later purposes, it is necessary to define the notion of the ‘Langlands dual’ of each seed. For a motivation, consult e.g. the ‘cluster linear algebra’ in [FG09, §2.3]. Here I just formulate the definitions that we will use, without explaining the reasoning behind them.

**Definition 58.** *For a seed  $\Gamma = (\varepsilon, d, *)$  of any kind, define the Langlands dual seed  $\Gamma^\vee = (\varepsilon^\vee, d^\vee, *^\vee)$  as*

$$(5.15) \quad \varepsilon_{ij}^\vee := d_i d_j^{-1} \varepsilon_{ij}, \quad d_i^\vee := d_i^{-1}, \quad \forall i, j = 1, \dots, n.$$

In particular, note that  $\varepsilon_{ij}^\vee (d_j^\vee)^{-1} = d_i \varepsilon_{ij} \stackrel{(2.2)}{=} \widetilde{\varepsilon}_{ij}$  is skew-symmetric. Here I point out an unfortunate typo in [FG07, eq.(5)] and [FG09, end of §2.1], which says  $\varepsilon_{ij}^\vee = -\varepsilon_{ji}^\vee$ , as if the matrix  $\varepsilon^\vee$  is skew-symmetric. It should be corrected to

$$\varepsilon_{ij}^\vee = -\varepsilon_{ji},$$

which one can easily deduce from  $\varepsilon_{ij}^\vee = d_i d_j^{-1} \varepsilon_{ij}$  and the skew-symmetry of  $\widehat{\varepsilon}_{ij} = d_j^{-1} \varepsilon_{ij}$ ; in particular,  $\varepsilon^\vee$  is an integer matrix. The duality assignment  $\varepsilon \rightsquigarrow \varepsilon^\vee$  of exchange matrices commutes with matrix mutations, i.e.

$$(\varepsilon'_{ij})^\vee = (\varepsilon_{ij}^\vee)'$$



where the prime ' in the LHS means mutation  $\mu_k$  applied to the matrix  $\varepsilon$  of  $\Gamma$  and the one in the RHS means mutation  $\mu_k^\vee$  applied to the Langlands dual matrix  $\varepsilon^\vee$  of  $\Gamma^\vee$ , which follows the same formula as the one of  $\mu_k$  for the usual exchange matrix:

$$(\varepsilon_{ij}^\vee)' := \begin{cases} -\varepsilon_{ij}^\vee & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij}^\vee + \frac{1}{2}(|\varepsilon_{ik}^\vee| \varepsilon_{kj}^\vee + \varepsilon_{ik}^\vee |\varepsilon_{kj}^\vee|) & \text{otherwise.} \end{cases}$$

Meanwhile,  $(d_i^\vee)' = d_i^\vee$  for all  $i$ , under the mutation  $\mu_k^\vee$ . Declare that the cluster variables mutate as if  $(\varepsilon^\vee, d^\vee, *^\vee)$  were a  $\mathcal{D}$ -seed; that is, put  $\vee$  at each cluster variable and  $\varepsilon \cdot$  appearing in the formulas (2.7), (2.8), (2.9). I use the notation  $P_\sigma^\vee$  for a seed automorphism of a Langlands dual seed, whose definition is same as  $P_\sigma$  with  $\vee$  put everywhere appropriate. So, a Langlands dual seed of a (genuine) seed can be viewed a generalized seed; it is not exactly a seed in the sense of Def.2 because the skew-symmetrizer  $(d^\vee)$  does not consist of positive integers.

**Definition 59.** For any cluster transformation  $\mathbf{m}$  of (genuine) seeds, denote by  $\mathbf{m}^\vee$  the corresponding cluster transformation of Langlands dual seeds.

**Definition 60.** Let  $\Gamma = (\varepsilon, d, \{B_i, X_i\}_{i=1}^n)$  be a  $\mathcal{D}$ -seed, and  $\Gamma^\vee = (\varepsilon^\vee, d^\vee, \{B_i^\vee, X_i^\vee\}_{i=1}^n)$  be its Langlands dual  $\mathcal{D}$ -seed. The Langlands dual quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_{\Gamma^\vee}^{q^\vee}$ , for a formal quantum parameter  $q^\vee$ , is the free associative  $*$ -algebra over  $\mathbb{Z}[q^\vee, (q^\vee)^{-1}]$  generated by  $\mathbf{B}_i^\vee, \mathbf{X}_i^\vee, i = 1, \dots, n$ , and their inverses, mod out by the relations

$$\begin{aligned} (q_j^\vee)^{-\varepsilon_{ij}^\vee} \mathbf{X}_i^\vee \mathbf{X}_j^\vee &= (q_i^\vee)^{-\varepsilon_{ji}^\vee} \mathbf{X}_j^\vee \mathbf{X}_i^\vee, & \forall i, j \in \{1, \dots, n\}, \\ (q_i^\vee)^{-1} \mathbf{X}_i^\vee \mathbf{B}_i^\vee &= (q_i^\vee) \mathbf{B}_i^\vee \mathbf{X}_i^\vee, & \forall i = 1, \dots, n, \\ \mathbf{B}_i^\vee \mathbf{X}_j^\vee &= \mathbf{X}_j^\vee \mathbf{B}_i^\vee, & \text{whenever } i \neq j, \\ \mathbf{B}_i^\vee \mathbf{B}_j^\vee &= \mathbf{B}_j^\vee \mathbf{B}_i^\vee, & \forall i, j \in \{1, \dots, n\}, \end{aligned}$$

where

$$(5.16) \quad q_i^\vee := (q^\vee)^{1/d_i^\vee} = (q^\vee)^{d_i} \in \mathbb{Z}[q^\vee, (q^\vee)^{-1}], \quad \forall i = 1, \dots, n,$$

with the  $*$ -structure defined as the unique ring anti-homomorphism satisfying

$$*\mathbf{X}_i^\vee = \mathbf{X}_i^\vee, \quad *\mathbf{B}_i^\vee = \mathbf{B}_i^\vee, \quad \forall i \in \{1, \dots, n\}, \quad *q^\vee = (q^\vee)^{-1}.$$

Denote by  $\mathbb{D}_{\Gamma^\vee}^{q^\vee}$  the skew field of fractions of  $\mathbf{D}_{\Gamma^\vee}^{q^\vee}$ . For each  $i \in \{1, \dots, n\}$  define

$$\tilde{\mathbf{X}}_i^\vee := \mathbf{X}_i^\vee \prod_{j=1}^n (\mathbf{B}_j^\vee)^{\varepsilon_{ij}^\vee} \in \mathbf{D}_{\Gamma^\vee}^{q^\vee}.$$

**Definition 61** (quantum mutation map for Langlands dual). Let the two  $\mathcal{D}$ -seeds  $\Gamma = (\varepsilon, d, *)$  and  $\Gamma' = (\varepsilon', d', *')$  be related by the mutation along  $k$ , that is,  $\Gamma' = \mu_k(\Gamma)$ , so that the Langlands dual seeds  $\Gamma^\vee = (\varepsilon^\vee, d^\vee, *^\vee)$  and  $(\Gamma')^\vee = ((\varepsilon')^\vee, (d')^\vee, (*')^\vee)$  are related by  $(\Gamma')^\vee = \mu_k(\Gamma^\vee)$ . Define the map  $\mu_k^{q^\vee} : \mathbb{D}_{(\Gamma')^\vee}^{q^\vee} \rightarrow \mathbb{D}_{\Gamma^\vee}^{q^\vee}$  by

$$\mu_k^{q^\vee} := \mu_k^{\sharp q^\vee} \circ (\mu'_k)^\vee,$$

where  $\mu_k^{\sharp q^\vee}$  is the automorphism of  $\mathbb{D}_{\Gamma^\vee}^{q^\vee}$  given by the following formulas on generators

$$(5.17) \quad \mu_k^{\sharp q^\vee}(\mathbf{B}_i^\vee) = \begin{cases} \mathbf{B}_i^\vee & \text{if } i \neq k, \\ \mathbf{B}_k^\vee (1 + q_k^\vee \mathbf{X}_k^\vee) (1 + q_k^\vee \tilde{\mathbf{X}}_k^\vee)^{-1} & \text{if } i = k, \end{cases}$$

$$(5.18) \quad \mu_k^{\sharp q^\vee}(\mathbf{X}_i^\vee) = \mathbf{X}_i^\vee \prod_{r=1}^{|\varepsilon_{ik}^\vee|} (1 + ((q_k^\vee)^{\text{sgn}(-\varepsilon_{ik}^\vee)})^{2r-1} \mathbf{X}_k^\vee)^{\text{sgn}(-\varepsilon_{ik}^\vee)}, \quad \forall i \in \{1, \dots, n\},$$

and  $(\mu'_k)^\vee$  is induced by the map  $(\mu'_k)^\vee : \mathbf{D}_{(\Gamma')^\vee}^{q^\vee} \rightarrow \mathbf{D}_{\Gamma^\vee}^{q^\vee}$  given by

$$\begin{aligned} (\mu'_k)^\vee((\mathbf{B}_i')^\vee) &= \begin{cases} \mathbf{B}_i^\vee & \text{if } i \neq k, \\ (\mathbf{B}_k^\vee)^{-1} \prod_{j=1}^n (\mathbf{B}_j^\vee)^{[-\varepsilon_{kj}^\vee]_+} & \text{if } i = k, \end{cases} \\ (\mu'_k)^\vee((\mathbf{X}_i')^\vee) &= \begin{cases} (q_k^\vee)^{\varepsilon_{ik}^\vee [\varepsilon_{ik}^\vee]_+} \mathbf{X}_i^\vee (\mathbf{X}_k^\vee)^{[\varepsilon_{ik}^\vee]_+} & \text{if } i \neq k, \\ (\mathbf{X}_k^\vee)^{-1} & \text{if } i = k, \end{cases} \end{aligned}$$

on the generators of  $\mathbf{D}_{(\Gamma')^\vee}^{q^\vee}$ .

**Lemma 62.** Each quantum  $\mathcal{D}$ -space functor  $\eta^q$  (3.4) induces a corresponding well-defined contravariant functor for the Langlands dual, denoted by  $\eta^{q^\vee}$ . ■

**5.3. The Schwartz space for universally Laurent elements.** For a representation  $(\mathcal{H}_\Gamma, D_\Gamma, \pi_\Gamma^q)$  of  $\mathbf{D}_\Gamma^q$  and a representation  $(\mathcal{H}_{\Gamma'}, D_{\Gamma'}, \pi_{\Gamma'}^q)$  of  $\mathbf{D}_{\Gamma'}^q$ , one would like to find a unitary ‘intertwiner’ map  $\mathcal{H}_\Gamma \rightarrow \mathcal{H}_{\Gamma'}$  that ‘intertwines’ the two representations  $\pi_\Gamma^q$  and  $\pi_{\Gamma'}^q$  of the two quantum torus algebras which are related by a quantum map  $\mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_\Gamma^q$  given by a composition of quantum mutation maps and quantum permutation maps. More precisely, each cluster transformation  $\mathbf{m} \in \text{Hom}_{\widehat{\mathcal{G}}_{|\Gamma|}^{\mathcal{D}}}(\Gamma, \Gamma')$  from  $\Gamma$  to  $\Gamma'$  in the saturated cluster modular groupoid  $\widehat{\mathcal{G}}_{|\Gamma|}^{\mathcal{D}}$  gives rise to a map  $\eta^q(\mathbf{m}) : \mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_\Gamma^q$  (Def.35), for which we must find a corresponding intertwiner. However, the spaces  $D_\Gamma$  and  $D_{\Gamma'}$  are not what are preserved by the intertwiner, nor does the intertwiner intertwines all elements of  $\mathbf{D}_\Gamma^q$  and  $\mathbf{D}_{\Gamma'}^q$ . One trouble is that the image of  $\mathbf{D}_{\Gamma'}^q \subset \mathbb{D}_{\Gamma'}^q$  under the quantum map in  $\mathbb{D}_\Gamma^q$  does not lie inside  $\mathbf{D}_\Gamma^q$ ; so one must consider the intersection of  $\mathbf{D}_\Gamma^q$  and this image, for  $\pi_\Gamma^q$  is defined only for  $\mathbf{D}_\Gamma^q \subset \mathbb{D}_\Gamma^q$ . Such should be taken into account for all  $\Gamma'$  and all quantum maps  $\mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_\Gamma^q$ , leading to the following notion.

**Definition 63** (quantum ring of universally Laurent polynomials). *Let  $\mathcal{C}$  be an equivalence class of  $\mathcal{D}$ -seeds. For any  $\Gamma \in \mathcal{C}$ , the  $*$ -algebra of universally Laurent elements  $\mathbb{L}_\mathcal{C}^q$  is defined as the following subring of the skew field of fractions  $\mathbb{D}_\Gamma^q$  of the seed quantum  $\mathcal{D}$ -torus algebra  $\mathbf{D}_\Gamma^q$ :*

$$(5.19) \quad \mathbb{L}_\Gamma^q := \bigcap_{\Gamma' \in \mathcal{C}} \bigcap_{\mathbf{m} \in \text{Hom}_{\widehat{\mathcal{G}}_{|\Gamma|}^{\mathcal{D}}}(\Gamma, \Gamma')} \eta^q(\mathbf{m})(\mathbf{D}_{\Gamma'}^q) \subset \mathbb{D}_\Gamma^q.$$

As each  $\eta^q(\mathbf{m}) : \mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_\Gamma^q$  is a  $*$ -algebra homomorphism, one observes that (5.19) indeed defines a  $*$ -subring of  $\mathbb{D}_\Gamma^q$ . Also, this definition of  $\mathbb{L}_\Gamma^q$  is ‘independent’ of the choice of a  $\mathcal{D}$ -seed  $\Gamma$  in the equivalence class  $\mathcal{C}$ . Namely, pick any  $\Gamma' \in \mathcal{C}$  and  $\mathbf{m} \in \text{Hom}_{\widehat{\mathcal{G}}_{|\Gamma|}^{\mathcal{D}}}(\Gamma, \Gamma')$ , then  $\eta^q(\mathbf{m})$  provides a  $*$ -algebra isomorphism  $\mathbb{L}_{\Gamma'}^q \rightarrow \mathbb{L}_\Gamma^q$ ; one may verify this, as well as the ‘consistency’ of this identification, from  $\eta^q(\mathbf{m}_1) \circ \eta^q(\mathbf{m}_2) = \eta^q(\mathbf{m}_2 \circ \mathbf{m}_1)$ , which holds because  $\eta^q$  is a contravariant functor. I collectively denote all  $\mathbb{L}_\Gamma^q$  for  $\Gamma \in \mathcal{C}$  by  $\mathbb{L}_\mathcal{C}^q$ .

We now should think of a suitable dense subspace of  $\mathcal{H}_\Gamma$  that the action  $\pi_\Gamma^q$  of  $\mathbb{L}_\Gamma^q$  is defined on and that is ‘preserved’ by the intertwiner. However, there is one more aspect to be investigated before we do so, namely the ‘modular double’ phenomenon. That is, the known intertwiner happens to intertwine the actions of the Langlands modular dual counterparts (Def.58) as well, namely the actions  $\pi_{\Gamma^\vee}^q$  and  $\pi_{(\Gamma')^\vee}^q$  of  $\mathbb{L}_{\Gamma^\vee}^q$  and  $\mathbb{L}_{(\Gamma')^\vee}^q$  related by  $\eta^{q^\vee}(\mathbf{m}^\vee)$  (Def.59, Lem.62), where  $\Gamma^\vee$  and  $(\Gamma')^\vee$  are the Langlands dual  $\mathcal{D}$ -seeds of  $\Gamma$  and  $\Gamma'$  (Def.58), while the quantum parameter for the Langlands dual

$$q^\vee := e^{\pi\sqrt{-1}(1/\hbar)}$$

is obtained by applying the ‘modular transformation’  $\hbar \mapsto 1/\hbar$  to  $q = e^{\pi\sqrt{-1}\hbar}$ . Moreover, the action of  $\mathbb{L}_\Gamma^q$  via  $\pi_\Gamma^q$  commutes with that of  $\mathbb{L}_{\Gamma^\vee}^q$  via  $\pi_{\Gamma^\vee}^q$  on a dense subspace of  $\mathcal{H}_\Gamma \equiv \mathcal{H}_{\Gamma^\vee}$ . This allows to put together these two actions to form a representation of the bigger algebra

$$(5.20) \quad \mathbb{L}_\Gamma^{\hbar} := \mathbb{L}_\Gamma^q \otimes_{\mathbb{Z}} \mathbb{L}_{\Gamma^\vee}^{q^\vee}$$

called the ‘(Langlands) modular double’ of  $\mathbb{L}_\Gamma^q$ , making the representation ‘strongly’ irreducible, in the sense that there is no more operator that commutes with all the operators representing the elements of the algebra via the representation. As a matter of fact, (5.20) is not the full description of the Langlands modular double; we also impose the transcendental relations, saying that the  $\frac{1}{\hbar_i}$ -th power of each generator  $\mathbf{B}_i$  (resp.  $\mathbf{X}_i$ ) of  $\mathbb{L}_\Gamma^q$  must ‘coincide with’ the generator  $\mathbf{B}_i^\vee$  (resp.  $\mathbf{X}_i^\vee$ ) of  $\mathbb{L}_{\Gamma^\vee}^{q^\vee}$ , which makes sense when considering representations, as we shall see (see the definition of ‘modular double’ e.g. in [FK12] [F95]), and which determines the precise relationship between  $\pi_\Gamma^q$  and  $\pi_{\Gamma^\vee}^{q^\vee}$ .

Define the following operators on the dense subspace  $D_\Gamma$  of the Hilbert space  $\mathcal{H}_\Gamma$ :

$$(5.21) \quad (\widehat{\mathbf{x}}_i^\hbar)^\vee := \frac{1}{\hbar_i} \widehat{\mathbf{x}}_i^\hbar, \quad (\widehat{\mathbf{b}}_i^\hbar)^\vee := \frac{1}{\hbar_i} \widehat{\mathbf{b}}_i^\hbar, \quad \forall i = 1, \dots, n,$$

where  $\widehat{\mathbf{x}}_i^\hbar$  and  $\widehat{\mathbf{b}}_i^\hbar$  are in (5.12) or in (5.13) and  $\hbar_i = \hbar/d_i$ . Then one immediately observes from (5.2) that the Weyl relations hold: namely, for  $(\alpha)$  and  $(\beta)$  as in (5.1), the unitary operators

$$(\widehat{\mathbf{B}}_i^\vee)^{(\alpha)} := e^{\sqrt{-1}\alpha_i(\widehat{\mathbf{b}}_i^\hbar)^\vee}, \quad \widehat{\mathbf{X}}_i^{(\beta)} := e^{\sqrt{-1}\beta_i(\widehat{\mathbf{x}}_i^\hbar)^\vee}, \quad i = 1, \dots, n,$$

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‘modular double’ was first discovered in [F95].

a genuine modular transformation would be  $\hbar \rightarrow -1/\hbar$ ; so, in fact, it might be better to put  $q^\vee = e^{\pi\sqrt{-1}(-1/\hbar)}$  instead, but  $q^\vee = e^{\pi\sqrt{-1}(1/\hbar)}$  is what has been used in the literature.

satisfy

$$\left\{ \begin{array}{ll} e^{\pi\sqrt{-1}h_j^\vee\varepsilon_{ij}\beta_i\beta_j} (\widehat{\mathbf{X}}_i^\vee)^{(\beta)} (\widehat{\mathbf{X}}_j^\vee)^{(\beta)} &= e^{\pi\sqrt{-1}h_i^\vee\varepsilon_{ji}\beta_j\beta_i} (\widehat{\mathbf{X}}_j^\vee)^{(\beta)} (\widehat{\mathbf{X}}_i^\vee)^{(\beta)}, & \forall i, j \in \{1, \dots, n\}, \\ e^{\pi\sqrt{-1}h_i^\vee\beta_i\alpha_i} (\widehat{\mathbf{X}}_i^\vee)^{(\beta)} (\widehat{\mathbf{B}}_i^\vee)^{(\alpha)} &= e^{-\pi\sqrt{-1}h_i^\vee\alpha_i\beta_i} (\widehat{\mathbf{B}}_i^\vee)^{(\alpha)} (\widehat{\mathbf{X}}_i^\vee)^{(\beta)}, & \forall i \in \{1, \dots, n\}, \\ (\widehat{\mathbf{B}}_i^\vee)^{(\alpha)} (\widehat{\mathbf{X}}_j^\vee)^{(\beta)} &= (\widehat{\mathbf{X}}_j^\vee)^{(\beta)} (\widehat{\mathbf{B}}_i^\vee)^{(\alpha)}, & \text{whenever } i \neq j, \\ (\widehat{\mathbf{B}}_i^\vee)^{(\alpha)} (\widehat{\mathbf{B}}_j^\vee)^{(\alpha)} &= (\widehat{\mathbf{B}}_j^\vee)^{(\alpha)} (\widehat{\mathbf{B}}_i^\vee)^{(\alpha)}, & \forall i, j \in \{1, \dots, n\}, \end{array} \right.$$

because, for example, the  $e$ -power in the LHS of the first equation is  $\pi\sqrt{-1}\beta_i\beta_j$  times

$$h_i^{-1}h_j^{-1}(h_j\varepsilon_{ij}) = h_i^{-1}\varepsilon_{ij} = h^{-1}d_i\varepsilon_{ij} = h^{-1}d_j(d_id_j^{-1}\varepsilon_{ij}) = h_j^\vee\varepsilon_{ij}^\vee,$$

where we denote

$$h_i^\vee := h^\vee/d_i^\vee = 1/h_i \quad \text{for each } i = 1, \dots, n.$$

These relations imply the Heisenberg counterparts as an analog of (5.5), which are easier to grasp than the Weyl relations: on  $D_\Gamma$ ,

$$[(\widehat{\mathbf{x}}_i^h)^\vee, (\widehat{\mathbf{x}}_j^h)^\vee] = 2\pi\sqrt{-1}h_j^\vee\varepsilon_{ij}^\vee \cdot \text{id}, \quad [(\widehat{\mathbf{x}}_i^h)^\vee, (\widehat{\mathbf{b}}_j^h)^\vee] = 2\pi\sqrt{-1}h_i^\vee\delta_{i,j} \cdot \text{id}, \quad [(\widehat{\mathbf{b}}_i^h)^\vee, (\widehat{\mathbf{b}}_j^h)^\vee] = 0,$$

for all  $i, j \in \{1, \dots, n\}$ . Moreover, again using (5.5) one finds

$$(5.22) \quad [\widehat{\mathbf{x}}_i^h, (\widehat{\mathbf{x}}_j^h)^\vee] = 2\pi\sqrt{-1}\varepsilon_{ij} \cdot \text{id}, \quad [\widehat{\mathbf{x}}_i^h, (\widehat{\mathbf{b}}_j^h)^\vee] = 2\pi\sqrt{-1}\delta_{i,j} \cdot \text{id}, \quad [\widehat{\mathbf{b}}_i^h, (\widehat{\mathbf{b}}_j^h)^\vee] = 0,$$

for all  $i, j$ ; note that all the commutators in (5.22) are identity times a scalar in  $2\pi\sqrt{-1}\mathbb{Z}$ .

One can then check, e.g. by using the Stone-von Neumann theorem or by dealing with the explicit example we presented, that

- Lemma 64.** 1) the operators  $\pi_\Gamma^q(\mathbf{B}_i^{\pm 1}) := e^{\pm \widehat{\mathbf{b}}_i^h}$ ,  $\pi_\Gamma^q(\mathbf{X}_i^{\pm 1}) := e^{\pm \widehat{\mathbf{x}}_i^h}$ ,  $i = 1, \dots, n$ , satisfy the defining relations of the generators  $\mathbf{B}_i^{\pm 1}$ ,  $\mathbf{X}_i^{\pm 1}$ ,  $i = 1, \dots, n$ , of the algebra  $\mathbf{D}_\Gamma^q$  on  $D_\Gamma$ ,  
 2) the operators  $\pi_{\Gamma^\vee}^q((\mathbf{B}_i^\vee)^{\pm 1}) := e^{\pm (\widehat{\mathbf{b}}_i^h)^\vee}$ ,  $\pi_{\Gamma^\vee}^q((\mathbf{X}_i^\vee)^{\pm 1}) := e^{\pm (\widehat{\mathbf{x}}_i^h)^\vee}$ ,  $i = 1, \dots, n$ , preserve  $D_\Gamma$ , are essentially self-adjoint on  $D_\Gamma$ , and satisfy the defining relations of the generators  $(\mathbf{B}_i^\vee)^{\pm 1}$ ,  $(\mathbf{X}_i^\vee)^{\pm 1}$ ,  $i = 1, \dots, n$ , of the algebra  $\mathbf{D}_{\Gamma^\vee}^q$  on  $D_\Gamma$ ,  
 3) the actions of  $\mathbf{D}_\Gamma^q$  and  $\mathbf{D}_{\Gamma^\vee}^q$  on  $D_\Gamma$  via  $\pi_\Gamma^q$  and  $\pi_{\Gamma^\vee}^q$  commute, in the sense that

$$\pi_\Gamma^q(\mathbf{u}) \pi_{\Gamma^\vee}^q(\mathbf{v}) = \pi_{\Gamma^\vee}^q(\mathbf{v}) \pi_\Gamma^q(\mathbf{u}) \quad \text{on } D_\Gamma,$$

for all generators  $\mathbf{u}$  and  $\mathbf{v}$  (hence for all elements) of  $\mathbf{D}_\Gamma^q$  and  $\mathbf{D}_{\Gamma^\vee}^q$ .

- 4) One has

$$(5.23) \quad \pi_{\Gamma^\vee}^q(\mathbf{B}_i^\vee) = (\pi_\Gamma^q(\mathbf{B}_i))^{1/h_i}, \quad \pi_{\Gamma^\vee}^q(\mathbf{X}_i^\vee) = (\pi_\Gamma^q(\mathbf{X}_i))^{1/h_i}, \quad \forall i = 1, \dots, n,$$

where the right hand sides are defined using the functional calculus of the unique self-adjoint extensions of  $\pi_\Gamma^q(\mathbf{B}_i)$  and  $\pi_\Gamma^q(\mathbf{X}_i)$ .

In particular, 3) can be formally proved by using the Baker-Campbell-Hausdorff (BCH) formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])+\dots}$$

and (5.22). Note that 4) is the ‘transcendental’ relation mentioned earlier. We could interpret the above 1)–4) as having a ‘positive integrable  $*$ -representation of the Langlands modular double algebra  $\mathbf{L}_\Gamma^h$ ’. Let us refer to this representation as

$$(5.24) \quad \pi_\Gamma^h := \pi_\Gamma^q \otimes \pi_{\Gamma^\vee}^q,$$

for convenience, by a slight abuse of notation.

We finally define the sought-for dense subspace  $\mathcal{S}_\Gamma^h$  of  $\mathcal{H}_\Gamma$ , which would serve as a maximal common domain for the operators representing the elements of  $\mathbf{L}_\Gamma^h$  via  $\pi_\Gamma^h$ , and which would be ‘preserved’ under sought-for intertwiners.

**Definition 65** (the Schwartz space of Fock and Goncharov; see e.g. [FG09, Def.5.2]). For a  $\mathcal{D}$ -seed  $\Gamma$ , define the seed  $\mathcal{D}$ -Schwartz space  $\mathcal{S}_\Gamma^h$  as

$$(5.25) \quad \mathcal{S}_\Gamma^h := \left\{ f \in \mathcal{H}_\Gamma \mid \begin{array}{l} \text{for each } \mathbf{u} \in \mathbf{L}_\Gamma^h \text{ (5.20), the linear functional } v \mapsto \langle \pi_\Gamma^h(\mathbf{u})v, f \rangle_{\mathcal{H}_\Gamma} \\ \text{is a continuous functional on } D_\Gamma \text{ (5.8)} \end{array} \right\}$$

Let  $f \in \mathcal{S}_\Gamma^h$ . Then, for each  $\mathbf{u} \in \mathbf{L}_\Gamma^h$ , how do we define  $\pi_\Gamma^h(\mathbf{u})f$ ? From the definition, the linear functional  $v \mapsto \langle \pi_\Gamma^h(*\mathbf{u})v, f \rangle_{\mathcal{H}_\Gamma}$  on  $D_\Gamma$  is continuous, because  $*\mathbf{u} \in \mathbf{L}_\Gamma^h$ . Then, since  $D_\Gamma$  is dense in  $\mathcal{H}_\Gamma$ , by Riesz Representation Theorem (or Riesz Lemma), there exists a unique vector  $g \in \mathcal{H}_\Gamma$  ‘representing’ this functional, i.e.  $\langle \pi_\Gamma^h(*\mathbf{u})v, f \rangle_{\mathcal{H}_\Gamma} = \langle v, g \rangle_{\mathcal{H}_\Gamma}$  for all  $v \in D_\Gamma$ . We now declare  $\pi_\Gamma^h(\mathbf{u})f$  to be this  $g$ .

Another crucial aspect of the Schwartz space  $\mathcal{S}_\Gamma^h$  is the Fréchet topology on it [G08] [FG09], given by the countable family of seminorms  $\rho_{\mathbf{u}}$  for  $\mathbf{u}$  running through a basis in  $\mathbf{L}_\Gamma^h$ , where

$$\rho_{\mathbf{u}}(v) := \|\pi_\Gamma^h(\mathbf{u})v\|_{\mathcal{H}_\Gamma}, \quad \forall v \in \mathcal{S}_\Gamma^h.$$

Although their subspaces  $\mathcal{S}_\mathbf{i}$  and  $W_\mathbf{i}$  of the Hilbert space  $L^2(\mathcal{A}_\mathbf{i}^+) \equiv L^2(\mathbb{R}^n)$  are different from our subspaces  $\mathcal{S}_\Gamma^h$  and  $D_\Gamma$  of  $\mathcal{H}_\Gamma \equiv L^2(\mathbb{R}^n)$ , I believe a same proof as the one for [FG09, Prop.5.8] would give the following, which can be interpreted as saying that the algebra of operators  $\pi_\Gamma^h(\mathbf{L}_\Gamma^h)$  acting on the dense subspace  $\mathcal{S}_\Gamma^h$  of the Hilbert space  $\mathcal{H}_\Gamma$  is essentially self-adjoint, as mentioned by Fock-Goncharov [FG09].

**Proposition 66.** *The space  $D_\Gamma$  is dense in the Schwartz space  $\mathcal{S}_\Gamma^h$  in the Fréchet topology.* ■

**5.4. Unitary intertwiners as quantum mutation operators, and the groupoid formulation of representations.** Here I shall finally say what an ‘intertwiner’ is precisely, and subsequently present a groupoid formulation of representations of (algebraic) quantum cluster  $\mathcal{D}$ -varieties. First, recall from Definitions 48 and 49 the notion of a weakly irreducible positive  $*$ -representation  $(\mathcal{H}_\Gamma, D_\Gamma, \pi_\Gamma^q)$  of a  $\mathcal{D}$ -seed  $\Gamma$ , where  $q = e^{\pi\sqrt{-1}h}$  with  $h \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ ; the datum  $\pi_\Gamma^q$  consists of essentially self-adjoint operators  $\widehat{\mathbf{b}}_{\Gamma,i}^h, \widehat{\mathbf{x}}_{\Gamma,i}^h$ ,  $i = 1, \dots, n$ , on a dense subspace  $D_\Gamma$  of a complex Hilbert space  $\mathcal{H}_\Gamma$ , satisfying the ‘Weyl relations’ as in Def.48.1). This entails some subsequent notions: the ‘tilde’ operators  $\widehat{\mathbf{x}}_{\Gamma,i}^h$  (5.6),  $i = 1, \dots, n$ , the ‘checked’ operators  $(\widehat{\mathbf{b}}_{\Gamma,i}^h)^\vee, (\widehat{\mathbf{x}}_{\Gamma,i}^h)^\vee$  (5.21),  $i = 1, \dots, n$ , the representation  $\pi_\Gamma^h$  of the algebra  $\mathbf{L}_\Gamma^h$  (5.20) of universally Laurent elements of the Langlands modular double quantum  $\mathcal{D}$ -torus algebra, and the Schwartz space  $\mathcal{S}_\Gamma^h$  (5.25).

**Definition 67.** *Let  $\mathcal{C}$  be an equivalence class of  $\mathcal{D}$ -seeds. Let  $h \in \mathbb{R}_{>0} \setminus \mathbb{Q}$  be a quantum parameter, and let  $q = e^{\pi\sqrt{-1}h}$ . The category of representations of quantum cluster  $\mathcal{D}$ -variety  $\text{DRep}_\mathcal{C}^h$  associated to  $\mathcal{C}$  is a category whose objects are weakly irreducible positive  $*$ -representation  $(\mathcal{H}_\Gamma, D_\Gamma, \pi_\Gamma^q)$  with  $\Gamma$  in  $\mathcal{C}$ , and whose set of morphisms from  $(\mathcal{H}_\Gamma, D_\Gamma, \pi_\Gamma^q)$  to  $(\mathcal{H}_{\Gamma'}, D_{\Gamma'}, \pi_{\Gamma'}^q)$  is the set of all intertwiners between these two objects, where an intertwiner for a cluster transformation  $\mathbf{m} \in \text{Hom}_{\widehat{\mathcal{G}}_\mathcal{C}^\mathcal{D}}(\Gamma, \Gamma')$  is a unitary map  $\mathbf{K}_{\Gamma,\Gamma'}^h : \mathcal{H}_{\Gamma'} \rightarrow \mathcal{H}_\Gamma$  satisfying the conditions*

- (Sch 1)  $\mathbf{K}_{\Gamma,\Gamma'}^h(\mathcal{S}_{\Gamma'}^h) = \mathcal{S}_\Gamma^h$ ,  
 (Sch 2)  $\mathbf{K}_{\Gamma,\Gamma'}^h \pi_{\Gamma'}^h(\mathbf{u}')v = \pi_\Gamma^h(\eta^h(\mathbf{m})(\mathbf{u}'))\mathbf{K}_{\Gamma,\Gamma'}^h v$  for all  $v \in \mathcal{S}_{\Gamma'}$  and  $\mathbf{u}' \in \mathbf{L}_{\Gamma'}^h$ ,  
 where  $\eta^h(\mathbf{m}) : \mathbf{L}_{\Gamma'}^h \rightarrow \mathbf{L}_\Gamma^h$  is the restriction of

$$(5.26) \quad \eta^h(\mathbf{m}) := \eta^q(\mathbf{m}) \otimes \eta^{q^\vee}(\mathbf{m}^\vee) : \mathbb{D}_{\Gamma'}^q \otimes \mathbb{D}_{(\Gamma')^\vee}^{q^\vee} \rightarrow \mathbb{D}_\Gamma^q \otimes \mathbb{D}_{\Gamma^\vee}^{q^\vee}.$$

**Definition 68.** *Let  $\mathcal{C}$ ,  $h$  be as in Def.67. By a representation of quantum cluster  $\mathcal{D}$ -variety associated to  $\mathcal{C}$  we mean a contravariant projective functor*

$$(5.27) \quad \mathbf{K}^h : \text{the saturated cluster modular groupoid } \widehat{\mathcal{G}}_\mathcal{C}^\mathcal{D} \text{ (Def.21)} \longrightarrow \text{DRep}_\mathcal{C}^h$$

such that  $\mathbf{K}^h(\mathbf{m})$  is an intertwiner for  $\mathbf{m}$  (Def.67) for each morphism  $\mathbf{m}$  of  $\widehat{\mathcal{G}}_\mathcal{C}^\mathcal{D}$ , where a projective functor means that the composition of morphisms are preserved only up to constants: for morphisms  $\mathbf{m}_1, \mathbf{m}_2$  such that  $\mathbf{m}_1 \circ \mathbf{m}_2$  is well-defined, there is a constant  $c_{\mathbf{m}_1, \mathbf{m}_2} \in \text{U}(1) \subset \mathbb{C}^\times$  such that

$$\mathbf{K}^h(\mathbf{m}_1 \circ \mathbf{m}_2) = c_{\mathbf{m}_1, \mathbf{m}_2} \cdot \mathbf{K}^h(\mathbf{m}_2) \mathbf{K}^h(\mathbf{m}_1).$$

Recall from §2.3 that the groupoid  $\widehat{\mathcal{G}}_\mathcal{C}^\mathcal{D}$  can be presented as ‘generators and relations’, with generators being mutations  $\mu_k$  (for  $k \in \{1, \dots, n\}$ ), together with seed automorphisms  $P_\sigma$  (for  $\sigma$  a permutation of the set  $\{1, \dots, n\}$ ), and relations being the ones in Lemmas 16, 17, and 18. So, in order to give a construction of a representation  $\mathbf{K}^h$  of quantum cluster  $\mathcal{D}$ -variety associated to  $\mathcal{C}$ , it suffices to describe what  $\mathbf{K}^h(\mu_k)$  and  $\mathbf{K}^h(P_\sigma)$  are, and verify that they satisfy the relations (written in the reverse orders) up to constants. For  $\mathbf{K}^h(P_\sigma)$  we take

$$(5.28) \quad \mathbf{K}^h(P_\sigma) := \mathbf{P}_\sigma : \mathcal{H}_{P_\sigma(\Gamma)} \rightarrow \mathcal{H}_\Gamma,$$

given by

$$(\mathbf{P}_\sigma f)(a_1, \dots, a_n) := f(a_1, \dots, a_n), \quad \forall f \in \mathcal{H}_{P_\sigma(\Gamma)} = L^2(\mathbb{R}^n, da_{\sigma(1)} \cdots da_{\sigma(n)}).$$

This description of  $\mathbf{P}_\sigma$  calls for a few words. Here, elements of  $\mathcal{H}_{P_\sigma(\Gamma)}$  are functions in the variables  $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ , where the notation is such that the  $i$ -th argument is  $a_{\sigma(i)}$ . To see what is happening, it is wise to extend the definition

of  $\mathbf{P}_\sigma$  to the space of ‘all’ functions on  $\mathbb{R}^n$  in the variables  $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ , by the same formula. For example, the symbol  $f := a_{\sigma(i)}$ , regarded as a function on  $\mathbb{R}^n$  in the variables  $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ , is the function that spits out the  $i$ -th coordinate as its value (in the  $P_\sigma(\Gamma)$  world); one can write  $f(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = a_{\sigma(i)}$  explicitly. The above formula then says  $(\mathbf{P}_\sigma f)(a_1, \dots, a_n) = a_i$ ; so,  $\mathbf{P}_\sigma f$ , regarded as a function on  $\mathbb{R}^n$  in the variables  $a_1, \dots, a_n$ , is what spits out the  $i$ -th coordinate, hence can be conveniently wrote as  $a_i$  (in the  $\Gamma$  world). Hence

$$\mathbf{P}_\sigma a_{\sigma(i)} = a_i, \quad \forall i = 1, \dots, n.$$

The following two subsections §5.5 and §5.6 are devoted to the two main ingredients for the construction in §5.7 of the intertwiner  $\mathbf{K}^h(\mu_k)$  for a mutation  $\mu_k$ . The relations which must be satisfied by  $\mathbf{K}^h(\mu_k)$  and  $\mathbf{K}^h(P_\sigma)$  are the subject of the next section §6.

**5.5. Special affine shift operators.** Let  $(M, \mu)$  and  $(N, \nu)$  be some measure spaces, and let  $\phi : M \rightarrow N$  be an invertible measure-preserving map whose inverse is also measure-preserving. It induces a natural unitary isomorphism of Hilbert spaces  $L^2(M, d\mu) \rightarrow L^2(N, d\nu)$  given by  $f \mapsto (\phi^{-1})^* f = f \circ \phi^{-1}$ ; this simple observation is generally referred to as ‘Koopman’s Lemma’ [K31]. In the present subsection we define and investigate some special cases, when both measure spaces are the usual Euclidean space  $(\mathbb{R}^n, da_1 \dots da_n)$  and  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is of the following type.

**Definition 69.** Write each element of  $\mathbb{R}^n$  as a row vector  $\mathbf{a} = (a_1 \ a_2 \ \dots \ a_n)$  with real entries.

A bijective map  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called an special affine transformation if there exist some matrix  $\mathbf{c} = (c_{ij})_{i,j \in \{1, \dots, n\}} \in \text{SL}_\pm(n, \mathbb{R})$  and some  $\mathbf{t} \in \mathbb{R}^n$ , where

$$(5.29) \quad \text{SL}_\pm(n, \mathbb{R}) := \{ \mathbf{c} \in \text{GL}(n, \mathbb{R}) : |\det \mathbf{c}| = 1 \},$$

such that

$$(5.30) \quad \phi^{-1}(\mathbf{a}) = \mathbf{a} \mathbf{c} + \mathbf{t}, \quad \forall \mathbf{a} \in \mathbb{R}^n.$$

The notions and results in the present subsection are nothing fundamentally new, but as I have never seen them formulated as a special class of operators, I find it convenient to investigate them as I shall do here now.

**Lemma 70.** The above correspondence  $\phi \rightsquigarrow (\mathbf{c}, \mathbf{t})$  is a group isomorphism between the group of all special affine transformations of  $\mathbb{R}^n$  and the following semi-direct product group

$$(5.31) \quad \text{SL}_\pm(n, \mathbb{R}) \ltimes \mathbb{R}^n = \{ (\mathbf{c}, \mathbf{t}) \mid \mathbf{c} \in \text{SL}_\pm(n, \mathbb{R}), \ \mathbf{t} \in \mathbb{R}^n \},$$

whose multiplication is given by

$$(5.32) \quad (\mathbf{c}, \mathbf{t}) (\mathbf{c}', \mathbf{t}') = (\mathbf{c} \mathbf{c}', \mathbf{t} \mathbf{c}' + \mathbf{t}').$$

*Proof.* Suppose two special affine transformations  $\phi$  and  $\psi$  are given by  $\phi^{-1}(\mathbf{a}) = \mathbf{a} \mathbf{c} + \mathbf{t}$  and  $\psi^{-1}(\mathbf{a}) = \mathbf{a} \mathbf{c}' + \mathbf{t}'$ . Then  $(\phi \circ \psi)^{-1}(\mathbf{a}) = \psi^{-1}(\phi^{-1}(\mathbf{a})) = \phi^{-1}(\mathbf{a}) \mathbf{c}' + \mathbf{t}' = (\mathbf{a} \mathbf{c} + \mathbf{t}) \mathbf{c}' + \mathbf{t}' = \mathbf{a} \mathbf{c} \mathbf{c}' + \mathbf{t} \mathbf{c}' + \mathbf{t}'$ . And the identity special affine transformation corresponds to  $(\mathbf{id}, \mathbf{0}) \in \text{SL}_\pm(n, \mathbb{R}) \ltimes \mathbb{R}^n$ , where  $\mathbf{id}$  denotes the identity matrix and  $\mathbf{0} = (0 \ \dots \ 0)$ .  $\blacksquare$

Let us adapt the notation for the Hilbert space  $\mathcal{H}_\Gamma$  and its dense subspace  $D_\Gamma$ , defined in (5.7) and (5.8) in Def.50. Note that the  $\mathcal{D}$ -seed data  $\Gamma$ , except for its ‘rank’  $n$ , is not playing any role in the definitions of  $\mathcal{H}_\Gamma$  and  $D_\Gamma$ ; it does only when we consider multiple  $\mathcal{D}$ -seeds at the same time, which is not the case in the present subsection. So, in principle we can drop the subscript  $\Gamma$  for now, but I keep it in order to avoid any unnecessary confusion.

**Lemma 71.** For any special affine transformation  $\phi$ , the map  $f \mapsto (\phi^{-1})^* f = f \circ \phi^{-1}$  is a unitary isomorphism from the Hilbert space  $\mathcal{H}_\Gamma$  (5.7) to itself.  $\blacksquare$

**Definition 72.** For each special affine transformation  $\phi$ , given by  $\phi^{-1}(\mathbf{a}) = \mathbf{a} \mathbf{c} + \mathbf{t}$ , denote the unitary automorphism  $f \mapsto (\phi^{-1})^* f$  of the Hilbert space  $\mathcal{H}_\Gamma$  (5.7) by  $\mathbf{S}_{(\mathbf{c}, \mathbf{t})}$ , which we call a special affine shift operator.

More explicitly,  $\mathbf{S}_{(\mathbf{c}, \mathbf{t})}$  is given by

$$(5.33) \quad (\mathbf{S}_{(\mathbf{c}, \mathbf{t})} f)(\mathbf{a}) = f(\mathbf{a} \mathbf{c} + \mathbf{t}), \quad \forall f \in \mathcal{H}_\Gamma = L^2(\mathbb{R}^n, da_1 \dots da_n), \quad \forall \mathbf{a} \in \mathbb{R}^n.$$

**Lemma 73.** The correspondence  $(\mathbf{c}, \mathbf{t}) \rightsquigarrow \mathbf{S}_{(\mathbf{c}, \mathbf{t})}$  is a group homomorphism, i.e.

$$(5.34) \quad \mathbf{S}_{(\mathbf{c}, \mathbf{t})}(\mathbf{c}', \mathbf{t}') = \mathbf{S}_{(\mathbf{c}, \mathbf{t})} \mathbf{S}_{(\mathbf{c}', \mathbf{t}')}, \quad \mathbf{S}_{(\mathbf{id}, \mathbf{0})} = \text{id}_{\mathcal{H}_\Gamma},$$

and is injective.

*Proof.* Straightforward exercise.  $\blacksquare$

**Corollary 74.** *The special affine shift operators on  $\mathcal{H}_\Gamma$  (5.7) form a group, isomorphic to  $\mathrm{SL}_\pm(n, \mathbb{R}) \ltimes \mathbb{R}^n$ . ■*

From the definition (5.33) we immediately obtain the following observation, which becomes useful in the proof of the main result of the present paper:

**Lemma 75.** *If  $\mathbf{S}_{(\mathbf{c}, \mathbf{t})} = c \cdot \mathrm{id}_{\mathcal{H}_\Gamma}$  for some complex scalar  $c$ , then  $c = 1$ . ■*

The following lemma is also immediate:

**Lemma 76.**  $\mathbf{S}_{(\mathbf{c}, \mathbf{t})}(D_\Gamma) = D_\Gamma$ . ■

The permutation operator  $\widehat{\mathbf{P}}_\sigma$  associated to a permutation  $\sigma$  of  $\{1, \dots, n\}$ , which is given by  $(\widehat{\mathbf{P}}_\sigma f)(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ , is an example of a special affine shift operator. Another class of examples is given by the usual ‘shift operators  $\mathbf{S}_{(\mathrm{id}, \mathbf{t})}$ ’: for any  $\mathbf{t} \in \mathbb{R}^n$  observe

$$(5.35) \quad (\mathbf{S}_{(\mathrm{id}, \mathbf{t})} f)(\mathbf{a}) = f(\mathbf{a} + \mathbf{t}).$$

In view of the Taylor series expansion, one may expect that we could write

$$(5.36) \quad \mathbf{S}_{(\mathrm{id}, \mathbf{t})} = e^{\sum_{i=1}^n t_i \frac{\partial}{\partial a_i}},$$

which a priori makes sense only formally; see [K16l] for a rigorous treatment. From (5.34), (5.35), and (5.36), one gets the following (see [K16l] for a proof).

**Lemma 77.** *Let  $\mathbf{c} \in \mathrm{SL}_\pm(n, \mathbb{R})$  and consider  $\mathbf{S}_{(\mathbf{c}, \mathbf{0})} : \mathcal{H}_\Gamma \rightarrow \mathcal{H}_\Gamma$ . Then, for each  $i = 1, \dots, n$ ,*

$$(5.37) \quad \mathbf{S}_{(\mathbf{c}, \mathbf{0})} \widehat{\mathbf{p}}_i^h \mathbf{S}_{(\mathbf{c}, \mathbf{0})}^{-1} = \sum_{j=1}^n c^{ij} \widehat{\mathbf{p}}_j^h \quad \text{on } D_\Gamma, \quad \text{where } \mathbf{c}^{-1} = (c^{ij})_{i,j \in \{1, \dots, n\}}.$$

*This equality also holds when  $\widehat{\mathbf{p}}_i^h$  and  $\sum_{j=1}^n c^{ij} \widehat{\mathbf{p}}_j^h$  are replaced by their unique self-adjoint extensions, on the domain of the self-adjoint extension of  $\sum_{j=1}^n c^{ij} \widehat{\mathbf{p}}_j^h$ .*

*One also has for each  $i = 1, \dots, n$ ,*

$$(5.38) \quad \mathbf{S}_{(\mathbf{c}, \mathbf{0})} \widehat{\mathbf{q}}_i^h \mathbf{S}_{(\mathbf{c}, \mathbf{0})}^{-1} = \sum_{j=1}^n c_{ji} \widehat{\mathbf{q}}_j^h \quad \text{on } D_\Gamma, \quad \text{where } \mathbf{c} = (c_{ij})_{i,j \in \{1, \dots, n\}}. \quad \blacksquare$$

**5.6. Quantum dilogarithm functions.** First recall the ‘compact’ quantum dilogarithm of Faddeev-Kashaev [FK94] [F95].

**Definition 78** (the compact quantum dilogarithm). *For any complex number  $\mathbf{q}$  with  $|\mathbf{q}| < 1$  define a meromorphic function  $\Psi^\mathbf{q}(z)$  on  $\mathbb{C}$  as*

$$(5.39) \quad \Psi^\mathbf{q}(z) := \prod_{i=1}^{\infty} (1 + \mathbf{q}^{2i-1} z)^{-1} = \frac{1}{(1 + \mathbf{q}z)(1 + \mathbf{q}^3 z)(1 + \mathbf{q}^5 z) \dots}$$

**Lemma 79.** *The above infinite product absolutely converges for any  $z$  except at the poles  $z = \mathbf{q}^{-(2i-1)}$ ,  $i = 1, 2, 3, \dots$  ■*

**Lemma 80.** *One has the functional equation*

$$(5.40) \quad \Psi^\mathbf{q}(\mathbf{q}^2 z) = (1 + \mathbf{q}z) \Psi^\mathbf{q}(z). \quad \blacksquare$$

For the moment, regard  $\mathbf{q}$  as a formal parameter; then one can make sense of  $\Psi^\mathbf{q}(z)^{-1}$  as a formal power series in  $z$  with coefficients in  $\mathbb{Z}_{\geq 0}[\mathbf{q}] \subset \mathbb{Z}[\mathbf{q}]$ . Let  $N$  be as in Def.29, and assume that  $\mathbf{q}^{1/N}$  is a well-defined formal variable, so that  $\mathbf{q}_k := \mathbf{q}^{1/k} = (\mathbf{q}^{1/N})^{N/k}$  is well-defined. Then, one can verify that the automorphism  $\mu_k^\sharp$  of  $\mathbb{D}_\Gamma^\mathbf{q}$  in Def.32 can be realized as conjugation by a single expression  $\Psi^{\mathbf{q}_k}(\mathbf{X}_k) \Psi^{\mathbf{q}_k}(\widetilde{\mathbf{X}}_k)^{-1}$ ;

$$(5.41) \quad \mu_k^\sharp(u) = \Psi^{\mathbf{q}_k}(\mathbf{X}_k) \Psi^{\mathbf{q}_k}(\widetilde{\mathbf{X}}_k)^{-1} u \Psi^{\mathbf{q}_k}(\widetilde{\mathbf{X}}_k) \Psi^{\mathbf{q}_k}(\mathbf{X}_k)^{-1}, \quad \forall u \in \mathbb{D}_\Gamma^\mathbf{q}.$$

Fock and Goncharov even took this formal conjugation (5.41) as the definition of  $\mu_k^\sharp$ ; see [FG09, Def.3.1]. The RHS of (5.41) is not a priori an element of  $\mathbb{D}_\Gamma^\mathbf{q}$ , but one can check that it is, for each generator  $u$  of  $\mathbb{D}_\Gamma^\mathbf{q}$ . For example,

$$\begin{aligned} \Psi^{\mathbf{q}_k}(\mathbf{X}_k) \Psi^{\mathbf{q}_k}(\widetilde{\mathbf{X}}_k)^{-1} \mathbf{X}_i \Psi^{\mathbf{q}_k}(\widetilde{\mathbf{X}}_k) \Psi^{\mathbf{q}_k}(\mathbf{X}_k)^{-1} &= \Psi^{\mathbf{q}_k}(\mathbf{X}_k) \mathbf{X}_i \Psi^{\mathbf{q}_k}(\mathbf{X}_k)^{-1} \\ &= \mathbf{X}_i \Psi^{\mathbf{q}_k}(\mathbf{q}_k^{-2\varepsilon_{ik}} \mathbf{X}_k) \Psi^{\mathbf{q}_k}(\mathbf{X}_k)^{-1}, \end{aligned}$$

which becomes (3.3) after a successive application of (5.40) on the part  $\Psi^{\mathbf{q}_k}(\mathbf{q}_k^{-2\varepsilon_{ik}} \mathbf{X}_k) \Psi^{\mathbf{q}_k}(\mathbf{X}_k)^{-1}$ .

However, for (physical) quantization of cluster varieties one needs the case when  $|\mathbf{q}| = 1$ . Thus I recall the ‘non-compact’ quantum dilogarithm; among its many guises, here I introduce a version as used in [FG09]. The non-compact quantum dilogarithm can be thought of as a suitable limit of the compact quantum dilogarithm as  $|\mathbf{q}| \rightarrow 1$ . The naïve limit does not exist; only certain ratio of compact quantum dilogarithm does. Note first that if we write  $\mathbf{q} = e^{\pi\sqrt{-1}h}$  for  $h \in \mathbb{C}$ , then  $|\mathbf{q}| < 1$  if and only if  $\text{Im}(h) > 0$ , and  $|\mathbf{q}| = 1$  means  $\text{Im}(h) = 0$ . As  $z \mapsto -1/z$  is an example of a  $\text{PSL}(2, \mathbb{R})$  Möbius transformation of the upper half plane, we see that  $\text{Im}(h) > 0$  implies  $\text{Im}(-1/h) > 0$ . So, if we define  $\mathbf{q}^\vee := e^{\pi\sqrt{-1}/h}$  for  $\text{Im}(h) > 0$ , we have  $|1/\mathbf{q}^\vee| < 1$ . When we form the ratio  $\Psi^{\mathbf{q}}(e^z)/\Psi^{1/\mathbf{q}^\vee}(e^{z/h})$ , the poles and zeroes make appropriate cancellation, so that the limit as  $\text{Im}(h) \rightarrow 0$  exists; this would then coincide with the non-compact quantum dilogarithm  $\Phi^h(z)$  with  $h \in \mathbb{R}$ . Here I write a little bit stronger statement, included in the following summary of some of the basic properties of  $\Phi^h$ :

**Lemma 81.** *For any  $h \in \mathbb{C}$  with  $\text{Im}(h) \geq 0$  and  $\text{Re}(h) > 0$ , let*

$$\mathbf{q} := e^{\pi\sqrt{-1}h}, \quad \mathbf{q}^\vee := e^{\pi\sqrt{-1}/h}.$$

a) *The integral in the expression*

$$\Phi^h(z) = \exp\left(-\frac{1}{4} \int_{\Omega} \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi h p)}\right),$$

*where  $\Omega$  is a contour on the real line that avoids the origin via a small half circle above the origin, absolutely converges for  $z$  in the strip  $\text{Im}(z) < \pi(1 + \text{Re}(h))$ , yielding a non-vanishing complex analytic function  $\Phi^h(z)$  on this strip. Each of the functional equations*

$$(5.42) \quad \begin{cases} \Phi^h(z + 2\pi\sqrt{-1}h) &= (1 + \mathbf{q}e^z)\Phi^h(z) \\ \Phi^h(z + 2\pi\sqrt{-1}) &= (1 + \mathbf{q}^\vee e^{z/h})\Phi^h(z) \end{cases}$$

*holds when the two arguments of  $\Phi^h$  are in the strip. These functional equations let us to analytically continue  $\Phi^h$  to a meromorphic function on the whole plane  $\mathbb{C}$ , with*

$$\text{the set of zeros} = \{(2\ell + 1)\pi\sqrt{-1} + (2m + 1)\pi\sqrt{-1}h \mid \ell, m \in \mathbb{Z}_{\geq 0}\}, \quad \text{and}$$

$$\text{the set of poles} = \{-(2\ell + 1)\pi\sqrt{-1} - (2m + 1)\pi\sqrt{-1}h \mid \ell, m \in \mathbb{Z}_{\geq 0}\}.$$

b) (relationship between compact and non-compact) *When  $\text{Im}(h) > 0$ , one has the equality*

$$(5.43) \quad \frac{\Psi^{\mathbf{q}}(e^z)}{\Psi^{1/\mathbf{q}^\vee}(e^{z/h})} = \Phi^h(z),$$

*for any  $z$  that is not a pole of  $\Phi^h$ .*

c) *For any sequence  $\{h_n\}_{n=1,2,\dots}$  in  $\mathbb{C}$  with  $\text{Im}(h_n) > 0$  and  $\text{Re}(h_n) > 0$  that converges to  $\hbar \in \mathbb{R}_{>0}$ , one has  $\lim_{n \rightarrow \infty} \Phi^{h_n}(z) = \Phi^{\hbar}(z)$  for almost every  $z$ .*

d) *When  $h = \hbar \in \mathbb{R}_{>0}$ , every pole and zero of  $\Phi^h$  is simple if and only if  $h \notin \mathbb{Q}$ .*

e) ( $\hbar \leftrightarrow 1/\hbar$  duality) *When  $h = \hbar \in \mathbb{R}_{>0}$ , one has*

$$(5.44) \quad \Phi^{1/\hbar}(z/\hbar) = \Phi^{\hbar}(z).$$

f) (unitarity) *When  $h = \hbar \in \mathbb{R}_{>0}$ , one has*

$$|\Phi^{\hbar}(z)| = 1, \quad \forall z \in \mathbb{R}.$$

g) (involutivity) *One has*

$$(5.45) \quad \Phi^h(z)\Phi^h(-z) = c_h \exp\left(\frac{z^2}{4\pi\sqrt{-1}h}\right),$$

*where*

$$(5.46) \quad c_h := e^{-\frac{\pi\sqrt{-1}}{12}(h+h^{-1})} \in \mathbb{C}^\times.$$

*In particular,  $|c_{\hbar}| = 1$  for  $\hbar = h \in \mathbb{R}_{>0}$ .*

■

**Remark 82.** *In the definition of  $\Phi^h$ , one may use a contour that avoids the origin by a small circle below the origin. The resulting value  $\Phi^h(z)$  is same as before.*

Besides the ‘difference equations’ (5.40) and (5.42), the quantum dilogarithm functions satisfy a crucial equation called the ‘pentagon equation’.

**Proposition 83** (the pentagon equation for the compact quantum dilogarithm; [FK94]). *Let  $X, Y$  be formal variables satisfying  $XY = \mathbf{q}^2 YX$ . Then one has*

$$(5.47) \quad \Psi^{\mathbf{q}}(Y)^{-1} \Psi^{\mathbf{q}}(X)^{-1} = \Psi^{\mathbf{q}}(X)^{-1} \Psi^{\mathbf{q}}(\mathbf{q}XY)^{-1} \Psi^{\mathbf{q}}(Y)^{-1},$$

where each factor and each side is regarded as a formal power series in the variables  $X$  and  $Y$  with coefficients in  $\mathbb{Z}_{\geq 0}[\mathbf{q}] \subset \mathbb{Z}[\mathbf{q}]$ . ■

When  $X$  and  $Y$  are formal variables with  $XY = \mathbf{q}^2 YX$ , we declare  $X^{1/h}$  and  $Y^{1/h}$  to be the new formal variables that satisfy  $X^{1/h} Y^{1/h} = (\mathbf{q}^{1/\vee})^2 Y^{1/h} X^{1/h}$  and commute with  $X$  and  $Y$ . Then from (5.47) for the compact quantum dilogarithms  $\Psi^{\mathbf{q}}$  and  $\Psi^{1/\mathbf{q}^\vee}$  one can formally deduce the corresponding pentagon equation for the non-compact quantum dilogarithm  $\Phi^h$ , which is given as ratio of these (5.43). It is best to formulate the pentagon equation of non-compact quantum dilogarithm in terms of an identity of unitary operators on a Hilbert space in the case when  $h = \hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ , so I postpone doing this until §6.2.

**5.7. Formulas for intertwiners.** I finally present Fock-Goncharov's formula for the intertwiner  $\mathbf{K}^h(\mu_k)$  for the mutation  $\mu_k$  from a  $\mathcal{D}$ -seed  $\Gamma$  to  $\Gamma'$ . As a way of remembering where  $\mu_k$  is being applied to and ends up with, we write

$$\mathbf{K}^h(\mu_k) = \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^h : \mathcal{H}_{\Gamma'} \rightarrow \mathcal{H}_{\Gamma}, \quad \text{when } \Gamma' = \mu_k(\Gamma).$$

**Definition 84** (the intertwiner for a mutation; [FG09, Def.5.1]). *Let  $\Gamma = (\varepsilon, d, \{B_i, X_i\}_{i=1}^n)$  be a  $\mathcal{D}$ -seed, and let  $\mu_k(\Gamma) = \Gamma' = (\varepsilon', d', \{B'_i, X'_i\}_{i=1}^n)$  for some  $k \in \{1, \dots, n\}$ . Let  $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ . We define the unitary map  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^h : \mathcal{H}_{\Gamma'} \rightarrow \mathcal{H}_{\Gamma}$  as*

$$(5.48) \quad \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^h := \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\#h} \circ \mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'},$$

where

$$\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\#h} := \Phi^{\hbar k}(\widehat{\mathbf{x}}_{\Gamma; k}^{\hbar}) \left( \Phi^{\hbar k}(\widehat{\mathbf{x}}_{\Gamma; k}^{\hbar}) \right)^{-1} : \mathcal{H}_{\Gamma} \rightarrow \mathcal{H}_{\Gamma}$$

is the composition of two unitary operators obtained by applying the functional calculus to the unique self-adjoint extensions of  $\widehat{\mathbf{x}}_{\Gamma; k}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma; k}^{\hbar}$  respectively which are defined either by (5.12) or by (5.13) for the unitary functions  $z \mapsto \Phi^{\hbar k}(z)$  and  $z \mapsto (\Phi^{\hbar k}(z))^{-1}$ , while

$$(5.49) \quad \mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} := \mathbf{S}(\mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'}, \mathbf{0}) \circ \mathbf{I}_{\Gamma \xrightarrow{k} \Gamma'} : \mathcal{H}_{\Gamma'} \rightarrow \mathcal{H}_{\Gamma},$$

where  $\mathbf{I}_{\Gamma \xrightarrow{k} \Gamma'} : \mathcal{H}_{\Gamma'} \rightarrow \mathcal{H}_{\Gamma}$  is induced by identifying each  $a'_i$  with  $a_i$ , i.e.

$$(5.50) \quad (\mathbf{I}_{\Gamma \xrightarrow{k} \Gamma'}, f)(a_1, \dots, a_n) := f(a'_1, \dots, a'_n), \quad \forall f \in \mathcal{H}_{\Gamma'} = L^2(\mathbb{R}^n, da'_1 \cdots da'_n),$$

and  $\mathbf{S}(\mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'}, \mathbf{0}) : \mathcal{H}_{\Gamma} \rightarrow \mathcal{H}_{\Gamma'}$  is defined as in (5.33) of §5.5, where  $\mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'} = (c_{ij})_{i,j \in \{1, \dots, n\}} \in \mathrm{SL}_{\pm}(n, \mathbb{R})$  is given by

$$(5.51) \quad c_{ii} = 1, \quad \forall i \neq k, \quad c_{kk} = -1, \quad c_{ik} = [-\varepsilon_{ki}]_+, \quad \forall i \neq k, \quad c_{ij} = 0 \quad \text{otherwise}.$$

So, as in eq.(73) of [FG09], one can view  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} : \mathcal{H}_{\Gamma'} \rightarrow \mathcal{H}_{\Gamma}$  as induced by the map

$$(5.52) \quad a'_i \mapsto \begin{cases} a_i & \text{if } i \neq k, \\ -a_k + \sum_{j=1}^n [-\varepsilon_{kj}]_+ a_j & \text{if } i = k. \end{cases}$$

From Lem.76 it follows that:

**Lemma 85.**  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}(D_{\Gamma'}) = D_{\Gamma}$ . ■

In order to verify that  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^h$  indeed intertwines the relevant actions, we begin with the conjugation action of  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}$  on the operators  $\widehat{\mathbf{b}}_{\Gamma'; i}^{\hbar}, \widehat{\mathbf{x}}_{\Gamma'; i}^{\hbar}, i = 1, \dots, n$ . For this, we apply Lem.77 to  $\mathbf{c} = \mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'}$ . From (5.51) one finds that  $\mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'}$  is its own inverse:

$$(5.53) \quad (\mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'})^{-1} = \mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'}.$$



**Lemma 86.** *Let  $\Gamma, \Gamma', k, \hbar$  be as in Def.84. One has*

$$(5.54) \quad \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \widehat{\mathbf{b}}_{\Gamma';i}^{\hbar} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = \widehat{\mathbf{b}}_{\Gamma;i}^{\hbar}, \quad \forall i \neq k,$$

$$(5.55) \quad \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \widehat{\mathbf{b}}_{\Gamma';k}^{\hbar} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = -\widehat{\mathbf{b}}_{\Gamma;k}^{\hbar} + \sum_{j=1}^n [-\varepsilon_{kj}]_+ \widehat{\mathbf{b}}_j^{\hbar},$$

as equalities on  $D_{\Gamma}$ , when the operators  $\widehat{\mathbf{b}}_{\Gamma';i}^{\hbar}$  and  $\widehat{\mathbf{b}}_{\Gamma;i}^{\hbar}$  on  $D_{\Gamma}$  are taken to be given by the ‘old’ formulas (5.13) or by the ‘new’ formulas (5.12). If  $\widehat{\mathbf{b}}_{\Gamma';i}^{\hbar}$  and  $\widehat{\mathbf{b}}_{\Gamma;i}^{\hbar}$  are replaced by their unique self-adjoint extensions, each equality still holds on the domain of the self-adjoint extension of either side of the equation.

*Proof.* For each  $i = 1, \dots, n$  observe

$$\begin{aligned} \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \widehat{\mathbf{q}}_{\Gamma';i}^{\hbar} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} &= (\mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})} \circ \mathbf{I}_{\Gamma \rightarrow \Gamma'}) (\widehat{\mathbf{q}}_{\Gamma';i}^{\hbar}) (\mathbf{I}_{\Gamma \rightarrow \Gamma'}^{-1} \circ \mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})}^{-1}) \\ &= \mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})} (\widehat{\mathbf{q}}_{\Gamma';i}^{\hbar}) \mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})}^{-1} \stackrel{\vee}{=} \begin{cases} \widehat{\mathbf{q}}_{\Gamma';i}^{\hbar}, & i \neq k, \\ -\widehat{\mathbf{q}}_{\Gamma;k}^{\hbar} + \sum_{j=1}^n [-\varepsilon_{kj}]_+ \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar}, & i = k, \end{cases} \end{aligned}$$

where the checked equality is from (5.38), (5.51), and  $\varepsilon_{kk} = 0$ . From (5.12), (5.13) we get the desired result.  $\blacksquare$

**Lemma 87.** *Let  $\Gamma, \Gamma', k, \hbar$  be as in Def.84. One has*

$$(5.56) \quad \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \widehat{\mathbf{x}}_{\Gamma';i}^{\hbar} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = \widehat{\mathbf{x}}_{\Gamma;i}^{\hbar} + [\varepsilon_{ik}]_+ \widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}, \quad \forall i \neq k,$$

$$(5.57) \quad \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \widehat{\mathbf{x}}_{\Gamma';i}^{\hbar} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = \widehat{\mathbf{x}}_{\Gamma;i}^{\hbar} + [\varepsilon_{ik}]_+ \widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}, \quad \forall i \neq k,$$

$$(5.58) \quad \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \widehat{\mathbf{x}}_{\Gamma';k}^{\hbar} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = -\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}, \quad \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \widehat{\mathbf{x}}_{\Gamma';k}^{\hbar} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = -\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar},$$

as equalities on  $D_{\Gamma}$ , when the operators  $\widehat{\mathbf{x}}_{\Gamma';i}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma;i}^{\hbar}$  are taken to be given by the ‘old’ formulas (5.13). These equalities do NOT hold when the operators are taken from the ‘new’ formulas (5.12). If  $\widehat{\mathbf{x}}_{\Gamma';i}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma;i}^{\hbar}$  are replaced by their unique self-adjoint extensions, each equality still holds on the domain of the self-adjoint extension of either side of the equation.

*Proof.* First, take the ‘old’ representation (5.13); then one writes for each  $i = 1, \dots, n$ ,

$$(5.59) \quad \begin{cases} (\widehat{\mathbf{x}}_{\Gamma;i}^{\hbar})^{\text{old}} = \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma;i}^{\hbar} - \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar}, & (\widehat{\mathbf{x}}_{\Gamma;i}^{\hbar})^{\text{old}} = \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma;i}^{\hbar} + \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar}, \\ (\widehat{\mathbf{x}}_{\Gamma';i}^{\hbar})^{\text{old}} = \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma';i}^{\hbar} - \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{q}}_{\Gamma';j}^{\hbar}, & (\widehat{\mathbf{x}}_{\Gamma';i}^{\hbar})^{\text{old}} = \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma';i}^{\hbar} + \sum_{j=1}^n \varepsilon'_{ij} \widehat{\mathbf{q}}_{\Gamma';j}^{\hbar}, \end{cases}$$

where  $\widehat{\mathbf{p}}_{\Gamma;i}^{\hbar}$  and  $\widehat{\mathbf{q}}_{\Gamma;j}^{\hbar}$  are defined on  $D_{\Gamma} \subset \mathcal{H}_{\Gamma}$  by (5.10), while  $\widehat{\mathbf{p}}_{\Gamma';i}^{\hbar}$  and  $\widehat{\mathbf{q}}_{\Gamma';j}^{\hbar}$  are defined on  $D_{\Gamma'} \subset \mathcal{H}_{\Gamma'}$  by (5.10) with  $a_i, \frac{\partial}{\partial a_i}, d_i$  replaced by the primed versions  $a'_i, \frac{\partial}{\partial a'_i}, d'_i = d_i$ . First, observe

$$\begin{aligned} \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \left( \frac{d_k^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma';k}^{\hbar} \mp \sum_{j=1}^n \varepsilon'_{kj} \widehat{\mathbf{q}}_{\Gamma';j}^{\hbar} \right) (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} &= \mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})} \left( \frac{d_k^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma;k}^{\hbar} \mp \sum_{j=1}^n (\varepsilon'_{kj}) \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar} \right) \mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})}^{-1} \\ &\stackrel{\vee}{=} \frac{d_k^{-1}}{2} (-\widehat{\mathbf{p}}_{\Gamma;k}^{\hbar}) \mp \sum_{j \neq k} (-\varepsilon_{kj}) \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar} = - \left( \frac{d_k^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma;k}^{\hbar} \mp \sum_{j=1}^n \varepsilon_{kj} \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar} \right), \end{aligned}$$

where in the checked equality we used (5.37), (5.38), (5.53), (5.51),  $\varepsilon_{kk} = 0$ , and  $\varepsilon'_{kj} = -\varepsilon_{kj}$ . In view of (5.59), we can translate this result into the following:

$$\mathbf{K}'_{\Gamma \rightarrow \Gamma'} (\widehat{\mathbf{x}}_{\Gamma';k}^{\hbar})^{\text{old}} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = -(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{\text{old}}, \quad \mathbf{K}'_{\Gamma \rightarrow \Gamma'} (\widehat{\mathbf{x}}_{\Gamma';k}^{\hbar})^{\text{old}} (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} = -(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{\text{old}},$$

which is the desired eq.(5.58) for the old representation.

For each  $i \neq k$  observe

$$(5.60) \quad \begin{aligned} \mathbf{K}'_{\Gamma \rightarrow \Gamma'} \left( \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma';i}^{\hbar} \mp \sum_{j=1}^n \varepsilon'_{ij} \widehat{\mathbf{q}}_{\Gamma';j}^{\hbar} \right) (\mathbf{K}'_{\Gamma \rightarrow \Gamma'})^{-1} &= \mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})} \left( \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma;i}^{\hbar} \mp \sum_{j=1}^n \varepsilon'_{ij} \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar} \right) \mathbf{S}_{(\mathbf{c}_{\Gamma \rightarrow \Gamma'}, \mathbf{o})}^{-1} \\ &= \frac{d_i^{-1}}{2} (\widehat{\mathbf{p}}_{\Gamma;i}^{\hbar} + [-\varepsilon_{ki}]_+ \widehat{\mathbf{p}}_{\Gamma;k}^{\hbar}) \mp \sum_{j \neq k} \varepsilon'_{ij} \widehat{\mathbf{q}}_{\Gamma;j}^{\hbar} \mp \varepsilon'_{ik} \left( -\widehat{\mathbf{q}}_{\Gamma;k}^{\hbar} + \sum_{\ell \neq k} [-\varepsilon_{k\ell}]_+ \widehat{\mathbf{q}}_{\Gamma;\ell}^{\hbar} \right), \end{aligned}$$

using (5.37), (5.38), (5.53), (5.51). Note that  $[-\varepsilon_{ki}]_+ \frac{d_k}{d_i} \stackrel{(2.1)}{=} [-\widehat{\varepsilon}_{ki} d_k]_+ = [\widehat{\varepsilon}_{ik} d_k]_+ \stackrel{(2.1)}{=} [\varepsilon_{ik}]_+$ . We use the matrix mutation formula (2.5)  $\varepsilon'_{ij} = \varepsilon_{ij} + \frac{1}{2}(|\varepsilon_{ik}| \varepsilon_{kj} + \varepsilon_{ik} |\varepsilon_{kj}|)$  and  $\varepsilon'_{ik} = -\varepsilon_{ik}$ . Hence

$$(5.60) = \left( \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma; i}^h \mp \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{q}}_{\Gamma; j}^h \right) + \left( \frac{d_k^{-1}}{2} [\varepsilon_{ik}]_+ \widehat{\mathbf{p}}_{\Gamma; k}^h \mp \sum_{j \neq k} \underbrace{\left( \frac{|\varepsilon_{ik}| \varepsilon_{kj} + \varepsilon_{ik} |\varepsilon_{kj}|}{2} - \varepsilon_{ik} [-\varepsilon_{kj}]_+ \right)}_{\varepsilon_{kj}} \widehat{\mathbf{q}}_{\Gamma; j}^h \right)$$

Putting  $|\varepsilon_{ik}| = 2[\varepsilon_{ik}]_+ - \varepsilon_{ik}$  and  $|\varepsilon_{kj}| = 2[-\varepsilon_{kj}]_+ + \varepsilon_{kj}$  makes the underbraced part to

$$\frac{2[\varepsilon_{ik}]_+ - \varepsilon_{ik}}{2} \varepsilon_{kj} + \varepsilon_{ik} (2[-\varepsilon_{kj}]_+ + \varepsilon_{kj}) - \varepsilon_{ik} [-\varepsilon_{kj}]_+ = [\varepsilon_{ik}]_+ \varepsilon_{kj}.$$

Thus, keeping in mind  $\varepsilon_{kk} = 0$ , we get

$$(5.60) = \left( \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma; i}^h \mp \sum_{j=1}^n \varepsilon_{ij} \widehat{\mathbf{q}}_{\Gamma; j}^h \right) + [\varepsilon_{ik}]_+ \left( \frac{d_k^{-1}}{2} \widehat{\mathbf{p}}_{\Gamma; k}^h \mp \sum_{j=1}^n \varepsilon_{kj} \widehat{\mathbf{q}}_{\Gamma; j}^h \right),$$

which is the desired eq.(5.56), (5.57) for the old representation.

On the other hand, let us try to use the new representation: in this case we have

$$(\widehat{\mathbf{x}}_{\Gamma; i}^h)^{\text{new}} = d_i^{-1} \widehat{\mathbf{p}}_{\Gamma; i}^h - \sum_{j=1}^n [\varepsilon_{ij}]_+ \widehat{\mathbf{q}}_{\Gamma; j}^h, \quad (\widehat{\mathbf{x}}_{\Gamma; i}^h)^{\text{new}} = d_i^{-1} \widehat{\mathbf{p}}_{\Gamma; i}^h - \sum_{j=1}^n [\varepsilon'_{kj}]_+ \widehat{\mathbf{q}}_{\Gamma; j}^h, \quad \forall i = 1, \dots, n.$$

We just check one case. Observe

$$\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} (\widehat{\mathbf{x}}_{\Gamma; k}^h)^{\text{new}} (\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'})^{-1} = \mathbf{S}_{(\mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'}, \mathbf{o})} \left( d_k^{-1} \widehat{\mathbf{p}}_{\Gamma; k}^h - \sum_{j=1}^n [\varepsilon'_{kj}]_+ \widehat{\mathbf{q}}_{\Gamma; j}^h \right) \mathbf{S}_{(\mathbf{c}_{\Gamma \xrightarrow{k} \Gamma'}, \mathbf{o})}^{-1} = -d_k^{-1} \widehat{\mathbf{p}}_{\Gamma; k}^h - \sum_{j \neq k} [-\varepsilon_{kj}]_+ \widehat{\mathbf{q}}_{\Gamma; j}^h,$$

where we used (5.37), (5.38), (5.53), (5.51), together with  $\varepsilon'_{kk} = 0$ . By inspection, one immediately finds that this does NOT equal  $-(\widehat{\mathbf{x}}_{\Gamma; k}^h)^{\text{new}}$ , unless  $[-\varepsilon_{kj}]_+ = -[\varepsilon_{kj}]_+$ ,  $\forall j \neq k$ , which implies  $\varepsilon_{kj} = 0$ ,  $\forall j \neq k$ , which is not always the case. In particular, if we require  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} (\widehat{\mathbf{x}}_{\Gamma; k}^h)^{\text{new}} (\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'})^{-1}$  to equal  $-(\widehat{\mathbf{x}}_{\Gamma; k}^h)^{\text{new}}$  for each  $k \in \{1, \dots, n\}$ , then we would have to have  $\varepsilon_{kj} = 0$  for all  $j, k \in \{1, \dots, n\}$ , which is undesirable. Eq.(5.56) and (5.57) for the new representation lead to more complicated equations for  $\varepsilon_{ij}$ 's, which do not always hold. ■

So, here I point out one mistake in [FG09]. The right identity in eq.(82) of Thm.5.6 in [FG09, §5.3] says  $\mathbf{K}' \widehat{A} w = \widehat{\mu'_k(A)} \mathbf{K}' w$  holds for all  $w \in W_{\Gamma'}$  and all  $A \in \mathbf{L}$ , where  $\widehat{A}$  and  $\widehat{\mu'_k(A)}$  stand for the operators for  $A$  and  $\mu'_k(A)$  corresponding to their 'new' representation, and  $\mu'_k$  is as in Def.32; in the proof they say that this is "straightforward". First, the right hand side of this identity must read  $\widehat{\mu'_k(A)} \mathbf{K}' w$ , which is an obvious minor typo. As mentioned earlier, their  $W_{\Gamma'}$  is a subspace of our  $D_{\Gamma'}$ . Notice that this asserted equation can be obtained from the exponentials of our (logarithmic) equations (5.56), (5.57), and (5.58). For example, the exponential of (5.58) would yield

$$\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} \pi_{\Gamma'}^h(\mathbf{X}_{\Gamma'; k})(\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'})^{-1} = \mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} e^{\widehat{\mathbf{x}}_{\Gamma'; k}^h} (\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'})^{-1} \stackrel{(5.58)}{=} e^{-\widehat{\mathbf{x}}_{\Gamma; k}^h} = \pi_{\Gamma'}^h(\mathbf{X}_{\Gamma; k}^{-1}) \stackrel{\text{Def.32}}{=} \pi_{\Gamma'}^h(\mu'_k(\mathbf{X}_{\Gamma'; k})).$$

However, recall from Lem.87 that (5.58) holds only for the 'old' representation, but not for the 'new' representation which is what is used in [FG09]. Likewise, the exponential of (5.56) would yield, for each  $i \neq k$ ,

$$\begin{aligned} \mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} \pi_{\Gamma'}^h(\mathbf{X}_{\Gamma'; i})(\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'})^{-1} &= \mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} e^{\widehat{\mathbf{x}}_{\Gamma'; i}^h} (\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'})^{-1} \stackrel{(5.56)}{=} e^{\widehat{\mathbf{x}}_{\Gamma; i}^h + [\varepsilon_{ik}]_+ \widehat{\mathbf{x}}_{\Gamma; k}^h} \\ &\stackrel{\vee}{=} q_k^{-\varepsilon_{ik} [\varepsilon_{ik}]_+} e^{\widehat{\mathbf{x}}_{\Gamma; i}^h} (e^{\widehat{\mathbf{x}}_{\Gamma; k}^h})^{[\varepsilon_{ik}]_+} = \pi_{\Gamma'}^h(q_k^{-\varepsilon_{ik} [\varepsilon_{ik}]_+} \mathbf{X}_i \mathbf{X}_k^{[\varepsilon_{ik}]_+}) \stackrel{\text{Def.32}}{=} \pi_{\Gamma'}^h(\mu'_k(\mathbf{X}_{\Gamma'; i})), \end{aligned}$$

where  $q_k = e^{\pi\sqrt{-1}\hbar_k} = e^{\pi\sqrt{-1}\hbar/d_k}$ ; the checked equality is from the BCH formula and  $[\widehat{\mathbf{x}}_{\Gamma; i}^h, \widehat{\mathbf{x}}_{\Gamma; k}^h] \stackrel{(5.5)}{=} 2\pi\sqrt{-1}\hbar_k \varepsilon_{ik} \cdot \text{id}$ . Similarly for (5.57). But again, these hold only for the 'old' representation, not the 'new' one.

Notice that in a previous paper [FG07] Fock and Goncharov used the 'old' representation along with the operator  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}$  given as in our (5.49). Then in [FG09] they decided to use the 'new' representation, mainly for aesthetic purpose in my opinion, but using the exactly same operator  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}$ . We just noted that  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}$  intertwine the 'old' operators correctly, but not the 'new' ones.

So the ‘new’ representation used in [FG09] must be retracted unless a corresponding correct operator  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}$  is found. Hence, from now on, we stick to their ‘old’ representation (5.13); the operators  $\widehat{\mathbf{b}}_{\Gamma;i}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma;i}^{\hbar}$  for each  $\mathcal{D}$ -seed  $\Gamma$  and  $i \in \{1, \dots, n\}$  would mean  $(\widehat{\mathbf{b}}_{\Gamma;i}^{\hbar})^{\text{old}}$  and  $(\widehat{\mathbf{x}}_{\Gamma;i}^{\hbar})^{\text{old}}$  respectively.

First, one can notice that the generators  $\mathbf{B}_{\Gamma';i}$ ,  $i = 1, \dots, n$ , as well as the ‘checked’ generators  $\mathbf{B}_{\Gamma';i}^{\vee}$ ,  $\mathbf{X}_{\Gamma';i}^{\vee}$ ,  $i = 1, \dots, n$ , are intertwined by  $\mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}$  in the way we want them to be, with respect to  $\mu'_k : \mathbb{D}_{\Gamma'}^q \rightarrow \mathbb{D}_{\Gamma}^q$  and  $(\mu'_k)^{\vee} : \mathbb{D}_{(\Gamma')^{\vee}}^q \rightarrow \mathbb{D}_{\Gamma^{\vee}}^q$ ; these two  $\mu'_k$  are collectively denoted by  $\mu'_k$  when applied on elements of  $\mathbf{L}_{\Gamma'}^{\hbar}$ , by abuse of notation. So we have

$$(5.61) \quad \mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'} \pi_{\Gamma'}^{\hbar}(\mathbf{u}') = \pi_{\Gamma}^{\hbar}(\mu'_k(\mathbf{u}')) \mathbf{K}'_{\Gamma \xrightarrow{k} \Gamma'}, \quad \forall \mathbf{u}' \in \mathbf{L}_{\Gamma'}^{\hbar}.$$

One may take this equality as an equality of operators  $D_{\Gamma'} \rightarrow D_{\Gamma}$ , and then take unique self-adjoint extensions of  $\pi_{\Gamma'}^{\hbar}(\mathbf{u}')$  and  $\pi_{\Gamma}^{\hbar}(\mu'_k(\mathbf{u}'))$ .

We refer the readers to [FG09, Thm.5.6] [K16] for the following conjugation action of  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\sharp \hbar}$ :

**Lemma 88.** *Let  $\Gamma, \Gamma', k, \hbar$  be as in Def.84. For each  $\mathbf{u} \in \mathbf{L}_{\Gamma}^{\hbar}$  one has*

$$(5.62) \quad \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\sharp \hbar} \pi_{\Gamma}^{\hbar}(\mathbf{u}) = \pi_{\Gamma}^{\hbar}(\mu_k^{\sharp \hbar}(\mathbf{u})) \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\sharp \hbar} \quad \text{on } D_{\Gamma},$$

where  $\mu_k^{\sharp \hbar} = \mu_k^{\sharp q} \otimes \mu_k^{\sharp q^{\vee}}$ .

From (5.48), (5.61), and (5.62), one would have

$$\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar} \pi_{\Gamma'}^{\hbar}(\mathbf{u}') (\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar})^{-1} = \pi_{\Gamma}^{\hbar}(\mu_k^{\hbar}(\mathbf{u}')), \quad \forall \mathbf{u}' \in \mathbf{L}_{\Gamma'}^{\hbar},$$

which holds when applied to a certain dense subspace as desired,  $\mu_k^{\hbar}$  denotes the restriction of

$$\mu_k^{\hbar} := \mu_k^q \otimes \mu_k^{q^{\vee}} : \mathbb{D}_{\Gamma'}^q \otimes \mathbb{D}_{(\Gamma')^{\vee}}^q \rightarrow \mathbb{D}_{\Gamma}^q \otimes \mathbb{D}_{\Gamma^{\vee}}^q$$

to a map  $\mathbf{L}_{\Gamma'}^{\hbar} \rightarrow \mathbf{L}_{\Gamma}^{\hbar}$ , just like (5.26).

In the next section, we shall concentrate on the ‘consistency’ relations of the unitary intertwiners  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar}$ ; for those I will provide rigorous proofs.

## 6. COMPUTATION OF OPERATOR IDENTITIES

In the present section, we always assume that

$$\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}.$$

**6.1. Operator identities as consequences of irreducibility.** In (5.28) and Def.84 of §5 we assigned unitary intertwiner maps  $\mathbf{K}^{\hbar}(P_{\sigma}) = \mathbf{P}_{\sigma}$  and  $\mathbf{K}^{\hbar}(\mu_k)$  to elementary morphisms  $P_{\sigma}$  and  $\mu_k$  of the saturated cluster modular groupoid  $\widehat{\mathcal{G}}_{\mathcal{C}}^{\mathcal{D}}$  (Def.21). In the present subsection we discuss the relations that they must satisfy: namely, the ones in Lemmas 16, 17, and 18, each written in the reverse order, up to a constant. The relations in Lem.17, i.e. the permutation identities, are easily checked to hold exactly, without multiplicative constants:

$$\mathbf{P}_{\gamma} \mathbf{P}_{\sigma} = \mathbf{P}_{\sigma \circ \gamma}, \quad \mathbf{P}_{\sigma^{-1}} \mathbf{K}^{\hbar}(\mu_k) \mathbf{P}_{\sigma} = \mathbf{K}^{\hbar}(\mu_{\sigma(k)}), \quad \mathbf{P}_{\text{id}} = \text{id}.$$

The way how Fock and Goncharov proves other relations in [FG09] is somewhat indirect. Suppose  $\mathbf{m}$  is a trivial morphism in  $\widehat{\mathcal{G}}_{\mathcal{C}}^{\mathcal{D}}$ , i.e. a morphism from an object  $\Gamma$  to itself, written as a particular sequence of mutations and seed automorphisms. Denote by  $\mathbf{K}^{\hbar}(\mathbf{m}) : \mathcal{H}_{\Gamma} \rightarrow \mathcal{H}_{\Gamma}$  the composition of the reversed sequence of intertwiners corresponding to these mutations and seed automorphisms. From the intertwining properties of  $\mathbf{K}^{\hbar}(\mu_k)$  and  $\mathbf{K}^{\hbar}(P_{\sigma})$  we have the following intertwining equality

$$\mathbf{K}^{\hbar}(\mathbf{m}) \pi_{\Gamma}^{\hbar}(\mathbf{u}) v = \pi_{\Gamma}^{\hbar}((\eta^{\hbar}(\mathbf{m}))(\mathbf{u})) \mathbf{K}^{\hbar}(\mathbf{m}) v, \quad \forall \mathbf{u} \in \mathbf{L}_{\Gamma}^{\hbar}, \quad \forall v \in \mathcal{S}_{\Gamma}^{\hbar},$$

where  $\eta^{\hbar}(\mathbf{m}) = \eta^q(\mathbf{m}) \otimes \eta^{q^{\vee}}(\mathbf{m}^{\vee})$ . As  $\mathbf{m}$  is a trivial morphism, Lem.37 tells us that both  $\eta^q(\mathbf{m})$  and  $\eta^{q^{\vee}}(\mathbf{m})$  are identity maps of algebras. Hence we have

$$\mathbf{K}^{\hbar}(\mathbf{m}) \pi_{\Gamma}^{\hbar}(\mathbf{u}) = \pi_{\Gamma}^{\hbar}(\mathbf{u}) \mathbf{K}^{\hbar}(\mathbf{m}), \quad \forall \mathbf{u} \in \mathbf{L}_{\Gamma}^{\hbar},$$

when applied to elements of  $v \in \mathcal{S}_{\Gamma}^{\hbar}$ . Fock and Goncharov [FG09] established that the representation  $\pi_{\Gamma}^{\hbar}$  of  $\mathbf{L}_{\Gamma}^{\hbar}$  (on  $\mathcal{S}_{\Gamma}^{\hbar}$ ) is ‘strongly irreducible’, in the sense that any bounded operator  $\mathcal{S}_{\Gamma}^{\hbar} \rightarrow \mathcal{S}_{\Gamma}^{\hbar}$  commuting with  $\pi_{\Gamma}^{\hbar}(\mathbf{u})$  for all

$\mathbf{u} \in \mathbf{L}_\Gamma^h$  is a scalar operator; they described such a situation as saying that the algebra  $\mathbf{L}_\Gamma^h$  is ‘big enough’. Anyhow, they proved:

**Theorem 89** ([FG09]). *For any trivial morphism  $\mathbf{m}$  in  $\widehat{\mathcal{G}}_\mathcal{C}^\mathcal{D}$  from  $\Gamma$  to itself written as a sequence of mutations and seed automorphisms, if  $\mathbf{K}^h(\mathbf{m})$  denotes the composition of the reversed sequence of intertwiners corresponding to the mutations and seed automorphisms constituting the sequence  $\mathbf{m}$ , then*

$$\mathbf{K}^h(\mathbf{m}) = c_{\mathbf{m}} \cdot \text{id}_{\mathcal{H}_\Gamma},$$

for some constant  $c_{\mathbf{m}} \in \text{U}(1) \subset \mathbb{C}^\times$ .  $\blacksquare$

**Corollary 90.** *The constructed intertwiners  $\mathbf{K}^h(\mu_k)$  and  $\mathbf{K}^h(P_\sigma)$  for the elementary cluster transformations induce a well-defined projective functor  $\mathbf{K}^h$  (5.27).  $\blacksquare$*

The constant  $c_{\mathbf{m}}$  is denoted by  $\lambda$  in [FG09, Thm.5.5], and is not precisely determined as an explicit number depending on  $h$ . A major goal of the present paper is to show by computation that this constant  $c_{\mathbf{m}}$  is always 1. For this, we write the result of the above theorem more explicitly in terms of the relations in Lemmas 16 and 18, like in [FG09, Thm.5.5] but even more concretely. I separate this into several cases.

**Proposition 91** (rank 1 identity; ‘twice-flip is identity’; [FG09]). *Let  $\Gamma = (\varepsilon, d, *)$  be a  $\mathcal{D}$ -seed,  $k \in \{1, \dots, n\}$ ,  $\Gamma' = \mu_k(\Gamma) = (\varepsilon', d', *)$ . Then  $\Gamma = \mu_k(\Gamma')$ , and there exists a constant  $c_{A_1} \in \text{U}(1) \subset \mathbb{C}^\times$  such that*

$$(6.1) \quad \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^h \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^h = c_{A_1} \cdot \text{id}_{\mathcal{H}_\Gamma}. \quad \blacksquare$$

Using the above Proposition, the  $(h+2)$ -gon relations can be made into the following forms.

**Proposition 92** ( $A_1 \times A_1$  identity; ‘commuting identity’; [FG09]). *Let  $\Gamma^{(0)} = (\varepsilon^{(0)}, d^{(0)}, *^{(0)})$  be a  $\mathcal{D}$ -seed, and assume that some two distinct  $i, j \in \{1, \dots, n\}$  satisfy*

$$(6.2) \quad \varepsilon_{ij}^{(0)} = 0 = \varepsilon_{ji}^{(0)}.$$

Let

$$\Gamma^{(1)} := \mu_i(\Gamma^{(0)}), \quad \Gamma^{(2)} := \mu_j(\Gamma^{(1)}), \quad \text{and} \quad \Gamma^{(3)} := \mu_j(\Gamma^{(0)}).$$

Then  $\Gamma^{(2)} = \mu_i(\Gamma^{(3)})$ , and there exists a constant  $c_{A_1 \times A_1} \in \text{U}(1) \subset \mathbb{C}^\times$  such that

$$(6.3) \quad \mathbf{K}_{\Gamma^{(0)} \xrightarrow{i} \Gamma^{(1)}}^h \mathbf{K}_{\Gamma^{(1)} \xrightarrow{j} \Gamma^{(2)}}^h = c_{A_1 \times A_1} \cdot \mathbf{K}_{\Gamma^{(0)} \xrightarrow{j} \Gamma^{(3)}}^h, \mathbf{K}_{\Gamma^{(3)} \xrightarrow{i} \Gamma^{(2)}}^h. \quad \blacksquare$$

**Proposition 93** ( $A_2$  identity; ‘pentagon identity’; [FG09]). *Let  $\Gamma^{(0)} = (\varepsilon^{(0)}, d^{(0)}, *^{(0)})$  be a  $\mathcal{D}$ -seed, and assume that some  $i, j \in \{1, \dots, n\}$  satisfy*

$$\begin{cases} \varepsilon_{ij}^{(0)} = 1, \\ \varepsilon_{ji}^{(0)} = -1, \end{cases} \quad \text{or} \quad \begin{cases} \varepsilon_{ij}^{(0)} = -1, \\ \varepsilon_{ji}^{(0)} = 1. \end{cases}$$

Let

$$\Gamma^{(1)} := \mu_i(\Gamma^{(0)}), \quad \Gamma^{(2)} := \mu_j(\Gamma^{(1)}), \quad \Gamma^{(3)} := \mu_i(\Gamma^{(2)}), \quad \text{and} \quad \Gamma^{(4)} := \mu_j(\Gamma^{(0)}), \quad \Gamma^{(5)} := \mu_i(\Gamma^{(4)}).$$

Then  $\Gamma^{(3)} = P_{(i,j)}(\Gamma^{(5)})$ , and there exists a constant  $c_{A_2} \in \text{U}(1) \subset \mathbb{C}^\times$  such that

$$(6.4) \quad \mathbf{K}_{\Gamma^{(0)} \xrightarrow{i} \Gamma^{(1)}}^h \mathbf{K}_{\Gamma^{(1)} \xrightarrow{j} \Gamma^{(2)}}^h \mathbf{K}_{\Gamma^{(2)} \xrightarrow{i} \Gamma^{(3)}}^h = c_{A_2} \cdot \mathbf{K}_{\Gamma^{(0)} \xrightarrow{j} \Gamma^{(4)}}^h \mathbf{K}_{\Gamma^{(4)} \xrightarrow{i} \Gamma^{(5)}}^h \mathbf{P}_{(i,j)},$$

as operators from  $\mathcal{H}_{\Gamma^{(3)}}$  to  $\mathcal{H}_{\Gamma^{(0)}}$ ; here  $\mathbf{P}_{(i,j)}$  is regarded as an operator from  $\mathcal{H}_{P_{(i,j)}(\Gamma^{(5)})} \equiv \mathcal{H}_{\Gamma^{(3)}}$  to  $\mathcal{H}_{\Gamma^{(5)}}$ .  $\blacksquare$

**Proposition 94** ( $B_2$  identity; ‘hexagon identity’; [FG09]). *Let  $\Gamma^{(0)} = (\varepsilon^{(0)}, d^{(0)}, *^{(0)})$  be a  $\mathcal{D}$ -seed, and assume that some  $i, j \in \{1, \dots, n\}$  satisfy*

$$\varepsilon_{ij}^{(0)} = -2\varepsilon_{ji}^{(0)}, \quad |\varepsilon_{ij}^{(0)}| = 2.$$

Let

$$\begin{aligned} \Gamma^{(1)} &:= \mu_i(\Gamma^{(0)}), & \Gamma^{(2)} &:= \mu_j(\Gamma^{(1)}), & \Gamma^{(3)} &:= \mu_i(\Gamma^{(2)}), & \text{and} \\ \Gamma^{(4)} &:= \mu_j(\Gamma^{(0)}), & \Gamma^{(5)} &:= \mu_i(\Gamma^{(4)}). \end{aligned}$$

Then  $\Gamma^{(3)} = \mu_j(\Gamma^{(5)})$ , and there exists a constant  $c_{B_2} \in \text{U}(1) \subset \mathbb{C}^\times$  such that

$$(6.5) \quad \mathbf{K}_{\Gamma^{(0)} \xrightarrow{i} \Gamma^{(1)}}^h \mathbf{K}_{\Gamma^{(1)} \xrightarrow{j} \Gamma^{(2)}}^h \mathbf{K}_{\Gamma^{(2)} \xrightarrow{i} \Gamma^{(3)}}^h = c_{B_2} \cdot \mathbf{K}_{\Gamma^{(0)} \xrightarrow{j} \Gamma^{(4)}}^h \mathbf{K}_{\Gamma^{(4)} \xrightarrow{i} \Gamma^{(5)}}^h \mathbf{K}_{\Gamma^{(5)} \xrightarrow{j} \Gamma^{(3)}}^h. \quad \blacksquare$$

**Proposition 95** ( $G_2$  identity; ‘octagon identity’; [FG09]). *Let  $\Gamma^{(0)} = (\varepsilon^{(0)}, d^{(0)}, *^{(0)})$  be a  $\mathcal{D}$ -seed, and assume that some  $i, j \in \{1, \dots, n\}$  satisfy*

$$\varepsilon_{ij}^{(0)} = -3\varepsilon_{ji}^{(0)}, \quad |\varepsilon_{ij}^{(0)}| = 3.$$

*Let*

$$\begin{aligned} \Gamma^{(1)} &:= \mu_i(\Gamma^{(0)}), & \Gamma^{(2)} &:= \mu_j(\Gamma^{(1)}), & \Gamma^{(3)} &:= \mu_i(\Gamma^{(2)}), & \Gamma^{(4)} &:= \mu_j(\Gamma^{(3)}), & \text{and} \\ \Gamma^{(5)} &:= \mu_j(\Gamma^{(0)}), & \Gamma^{(6)} &:= \mu_i(\Gamma^{(5)}), & \Gamma^{(7)} &:= \mu_j(\Gamma^{(6)}). \end{aligned}$$

*Then  $\Gamma^{(4)} = \mu_i(\Gamma^{(7)})$ , and there exists a constant  $c_{G_2} \in \mathbb{U}(1) \subset \mathbb{C}^\times$  such that*

$$\begin{aligned} (6.6) \quad & \mathbf{K}_{\Gamma^{(0)} \xrightarrow{i} \Gamma^{(1)}}^h \mathbf{K}_{\Gamma^{(1)} \xrightarrow{j} \Gamma^{(2)}}^h \mathbf{K}_{\Gamma^{(2)} \xrightarrow{i} \Gamma^{(3)}}^h \mathbf{K}_{\Gamma^{(3)} \xrightarrow{j} \Gamma^{(4)}}^h \\ &= c_{G_2} \cdot \mathbf{K}_{\Gamma^{(0)} \xrightarrow{j} \Gamma^{(5)}}^h \mathbf{K}_{\Gamma^{(5)} \xrightarrow{i} \Gamma^{(6)}}^h \mathbf{K}_{\Gamma^{(6)} \xrightarrow{j} \Gamma^{(7)}}^h \mathbf{K}_{\Gamma^{(7)} \xrightarrow{i} \Gamma^{(4)}}^h. \end{aligned} \quad \blacksquare$$

**Remark 96.** *The terminologies ‘hexagon’ and ‘octagon’ which I used for the last two equalities as analogs of ‘pentagon’ are probably not standard.*

**Remark 97.** *These constants depend only on the underlying ‘feed’ data  $(\varepsilon, d)$ , not on specific seeds  $\Gamma$ . So Prop.91–95 are slightly stronger than Thm.89.*

One cannot determine these constants  $c_{A_1}, c_{A_1 \times A_1}, c_{A_2}, c_{B_2}, c_{G_2}$  by means of the indirect argument of Fock and Goncharov. So we will make explicit computations about the unitary intertwining operators, and use some known operator identities involving the quantum dilogarithm function  $\Phi^h$ .

**6.2. Known operator identities.** In the current subsection, we establish some operator identities, including the pentagon equation for the non-compact quantum dilogarithm  $\Phi^h$  promised in §5.6.

Recall the identity  $\Phi^h(z)\Phi^h(-z) = c_h \cdot \exp(\frac{z^2}{4\pi\sqrt{-1}h})$  (5.45), where  $|c_h| = 1$ . As  $z \mapsto \Phi^h(z)$ ,  $z \mapsto \Phi^h(-z)$ , and  $z \mapsto \exp(\frac{z^2}{4\pi\sqrt{-1}h})$  are all unitary functions in the sense of Lem.46, the application of the functional calculus for a self-adjoint operator on these functions yield unitary operators.

**Lemma 98.** *Let  $T$  be a (densely-defined) self-adjoint operator on a Hilbert space  $V$ . Its functional calculus applied to the three unitary functions just mentioned yield three unitary operators  $\Phi^h(T)$ ,  $\Phi^h(-T)$ , and  $\exp(\frac{T^2}{4\pi\sqrt{-1}h})$ , satisfying*

$$(6.7) \quad \Phi^h(T)\Phi^h(-T) = c_h \exp\left(\frac{T^2}{4\pi\sqrt{-1}h}\right),$$

where  $c_h$  is as in (5.46).  $\blacksquare$

Proof of (6.7) is immediate from functional calculus.

**Lemma 99** (linear combination of Heisenberg operators). *Let  $P$  and  $Q$  be densely defined self-adjoint operators on a separable Hilbert space  $V$  satisfying the Weyl relations*

$$e^{\sqrt{-1}\alpha P} e^{\sqrt{-1}\beta Q} = e^{-2\pi\sqrt{-1}h\alpha\beta} e^{\sqrt{-1}\beta Q} e^{\sqrt{-1}\alpha P}, \quad \forall \alpha, \beta \in \mathbb{R},$$

corresponding to the Heisenberg relation

$$[P, Q] = 2\pi\sqrt{-1}h \cdot \text{id}$$

which makes sense on a dense subspace of  $V$ . Then there exists a dense subspace  $D$  of  $V$  such that

- 1) each of the restrictions  $\hat{\mathbf{p}}^h := P \upharpoonright D$  and  $\hat{\mathbf{q}}^h := Q \upharpoonright D$  preserves  $D$  and is essentially self-adjoint,
- 2) for each real numbers  $\alpha_0, \beta_0$ , the operator  $\alpha_0\hat{\mathbf{p}}^h + \beta_0\hat{\mathbf{q}}^h$  on  $D$  is essentially self-adjoint.

Denote by  $\alpha_0 P + \beta_0 Q$  the unique self-adjoint extension of  $\alpha_0\hat{\mathbf{p}}^h + \beta_0\hat{\mathbf{q}}^h$ .  $\blacksquare$

**Lemma 100.** *Let  $\Gamma$  be a  $\mathcal{D}$ -seed with  $n \geq 2$ , and  $D_\Gamma \subset \mathcal{H}_\Gamma = L^2(\mathbb{R}^n, da_1 \cdots da_n)$  be as in (5.7) and (5.8). Suppose  $\mathbf{o}_1$  and  $\mathbf{o}_2$  are operators on  $D_\Gamma$ , preserving  $D_\Gamma$ , given by  $\mathbb{R}$ -linear combinations of  $\hat{\mathbf{p}}_i^h, \hat{\mathbf{q}}_i^h$  (5.10),  $i = 1, \dots, n$ . Assume that none of the two is the zero operator, and that*

$$[\mathbf{o}_1, \mathbf{o}_2] = 0$$

holds as operators  $D_\Gamma \rightarrow D_\Gamma$ . If we denote by  $O_1$  and  $O_2$  the unique self-adjoint extensions of  $\mathbf{o}_1$  and  $\mathbf{o}_2$ , then for any unitary functions  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{C}$  (in the sense of Lem.46), the unitary operators  $f_1(O_1)$  and  $f_2(O_2)$  commute:

$$f_1(O_1)f_2(O_2) = f_2(O_2)f_1(O_1). \quad \blacksquare$$

Let us take the above two lemmas for granted; see [K16l] for proofs.

**Theorem 101** (the pentagon equation of the non-compact quantum dilogarithm). *Let  $P$ ,  $Q$ , and  $V$  be as in Lem.99. Then*

$$\Phi^{\hbar}(P) \Phi^{\hbar}(Q) = \Phi^{\hbar}(Q) \Phi^{\hbar}(Q + P) \Phi^{\hbar}(P).$$

As pointed out in [FG09], this pentagon equation was suggested in [F95] and proved in [FKV01], [W00], and [G08].

**Corollary 102.** *Let  $P$ ,  $Q$ , and  $V$  be as in Lem.99. Then*

$$(6.8) \quad \Phi^{\hbar}(P) \Phi^{\hbar}(Q) \Phi^{\hbar}(-P) = c_{\hbar} \Phi^{\hbar}(Q) \Phi^{\hbar}(Q + P) \exp\left(\frac{P^2}{4\pi\sqrt{-1}\hbar}\right). \quad \blacksquare$$

### 6.3. Rank 1 identity: ‘twice-flip’ is trivial.

**Proposition 103.**  $c_{A_1} = 1$ .

*Proof.* The LHS of (6.1) is

$$\begin{aligned} \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar} \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar} &= (\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\# \hbar} \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar}) (\mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\# \hbar} \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar}) \\ &= \Phi^{\hbar_k}(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}) \Phi^{\hbar_k}(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{-1} \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k} \Phi^{\hbar'_k}(\widehat{\mathbf{x}}_{\Gamma';k}^{\hbar}) \Phi^{\hbar'_k}(\widehat{\mathbf{x}}_{\Gamma';k}^{\hbar})^{-1} \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k} \\ &= \Phi^{\hbar_k}(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}) \Phi^{\hbar_k}(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{-1} \left( \Phi^{\hbar_k}(-\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}) \Phi^{\hbar_k}(-\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{-1} \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k} \right) \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k} \quad (\because \text{Lem.47, 87, } \hbar'_k = \hbar_k) \\ &\stackrel{\vee}{=} \underbrace{\Phi^{\hbar_k}(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}) \Phi^{\hbar_k}(-\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}) \Phi^{\hbar_k}(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{-1} \Phi^{\hbar_k}(-\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{-1}}_{\substack{(5.45) \\ = c_{\hbar_k} c_{\hbar_k}^{-1}}} \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k} \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k} \\ &= \underbrace{c_{\hbar_k} c_{\hbar_k}^{-1}}_{\substack{(5.45) \\ = c_{\hbar_k} c_{\hbar_k}^{-1}}} \exp\left(\frac{(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_k}\right) \exp\left(-\frac{(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_k}\right) \mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k} \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k}, \end{aligned}$$

where for the checked equality we used the fact that the two unitary operators  $\Phi^{\hbar_k}(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^{-1}$  and  $\Phi^{\hbar_k}(-\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})$ , where  $\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma';k}^{\hbar}$  here stand for the respective unique self-adjoint extensions, commute with each other, as follows from Lem.100 since  $[\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}, \widehat{\mathbf{x}}_{\Gamma';k}^{\hbar}] = 0$ .

We shall prove that the underbraced part is a special affine shift operator. If we identify  $\mathcal{H}_{\Gamma}$  and  $\mathcal{H}_{\Gamma'}$  by the natural relabelling isomorphisms  $\mathbf{I}_{\Gamma \xrightarrow{k} \Gamma'}$  and  $\mathbf{I}_{\Gamma' \xrightarrow{k} \Gamma}$  (5.50), we see from (5.49) that  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k}$  and  $\mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k}$  are special affine shift operators. Then one deduces from Cor.74 that  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k} \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k}$  is a special affine shift operator as well. As we already know from Prop.91 that it is a scalar operator, from Lem.75 we conclude that it is the identity operator. If one would like, one can prove without relying on Prop.91. We shall now compute what precise special affine shift operator that  $\exp\left(\frac{(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_k}\right) \exp\left(-\frac{(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_k}\right)$  equals to; this, together with the expressions for  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k}$  and  $\mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k}$  in terms of explicit special affine shift operators, allows one to compute  $\mathbf{K}_{\Gamma \xrightarrow{k} \Gamma'}^{\hbar_k} \mathbf{K}_{\Gamma' \xrightarrow{k} \Gamma}^{\hbar_k}$  as a single special affine shift operator, with the help of (5.34). One can directly prove by computation that this equals  $\mathbf{S}_{(\text{id}, 0)}$ .

So it remains to prove:

**Lemma 104.** *For any  $\mathcal{D}$ -seed  $\Gamma$ ,  $k \in \{1, \dots, n\}$ , and  $\hbar \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ , the unitary operator*

$$(6.9) \quad \exp\left(\frac{(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_k}\right) \exp\left(-\frac{(\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_k}\right)$$

on  $\mathcal{H}_{\Gamma}$ , whose two factors are obtained by applying the functional calculus to the unique self-adjoint extensions of the operators  $\widehat{\mathbf{x}}_{\Gamma;k}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma';k}^{\hbar}$  on  $D_{\Gamma}$  for the unitary functions  $z \mapsto \exp(\frac{z^2}{4\pi\sqrt{-1}\hbar_k})$  and  $z \mapsto \exp(-\frac{z^2}{4\pi\sqrt{-1}\hbar_k})$  respectively, coincides with the special affine shift operator  $\mathbf{S}_{(\mathbf{c}, 0)}$ , where  $\mathbf{c} = (c_{ij})_{i,j \in \{1, \dots, n\}}$  is given by

$$c_{ii} = 1, \quad \forall i = 1, \dots, n, \quad c_{ik} = -\varepsilon_{ki}, \quad \forall i \neq k, \quad c_{ij} = 0 \quad \text{otherwise.}$$

It is an easy exercise to find a rigorous proof, which is done in [K16]. Here I only give a formal argument, which gives one an idea on why it would hold. We do a formal manipulation as follows:

$$(6.9) = \exp \left( \frac{(\widehat{\mathbf{x}}_{\Gamma;k}^h)^2 - (\widehat{\mathbf{x}}_{\Gamma;k}^h)^2}{4\pi\sqrt{-1}\hbar_k} \right) = \exp \left( \frac{(\widehat{\mathbf{x}}_{\Gamma;k}^h + \widehat{\mathbf{x}}_{\Gamma;k}^h)(\widehat{\mathbf{x}}_{\Gamma;k}^h - \widehat{\mathbf{x}}_{\Gamma;k}^h)}{4\pi\sqrt{-1}\hbar_k} \right) \stackrel{(5.13),(5.14)}{=} \exp \left( \frac{-\widehat{\mathbf{p}}_k^h(\sum_{i \neq k} \varepsilon_{ki} \widehat{\mathbf{q}}_i^h)}{2\pi\sqrt{-1}\hbar} \right) =: T_k.$$

As  $\frac{\widehat{\mathbf{p}}_k^h}{2\pi\sqrt{-1}\hbar} \stackrel{(5.10)}{=} \frac{\partial}{\partial a_k}$  and  $\widehat{\mathbf{q}}_i^h = a_i$  on  $D_\Gamma$ , like we had in (5.36) and (5.35) one can observe that  $(T_k f)(a_1, \dots, a_k, \dots, a_n) = f(a_1, \dots, a_k - \sum_{i \neq k} \varepsilon_{ki} a_i, \dots, a_n)$ , giving the desired result.

#### 6.4. $A_1 \times A_1$ identity: the ‘commuting’ relation.

**Proposition 105.**  $c_{A_1 \times A_1} = 1$ .

*Proof.* The LHS of (6.3) is

$$\begin{aligned} \mathbf{K}_{\Gamma(0) \xrightarrow{i} \Gamma(1)}^h \mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^h &= (\mathbf{K}_{\Gamma(0) \xrightarrow{i} \Gamma(1)}^{\sharp h} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(1)}) (\mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^{\sharp h} \mathbf{K}'_{\Gamma(1) \xrightarrow{i} \Gamma(2)}) \\ &= \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} (\Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^h)^{-1} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)}) \\ &= \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} (\Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)}) \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)}, \quad (\because \text{Lem.47, 87, (6.2)}) \end{aligned}$$

where we are implicitly using the fact that each of  $d_i$  and  $d_j$  is the same for all  $\Gamma^{(\ell)}$ . Meanwhile, the RHS of (6.3) without the constant is

$$\begin{aligned} \mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(3)}^h \mathbf{K}_{\Gamma(3) \xrightarrow{i} \Gamma(2)}^h &= (\mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(3)}^{\sharp h} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(3)}) (\mathbf{K}_{\Gamma(3) \xrightarrow{i} \Gamma(2)}^{\sharp h} \mathbf{K}'_{\Gamma(3) \xrightarrow{j} \Gamma(2)}) \\ &= \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(3)} (\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(3);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(3);i}^h)^{-1} \mathbf{K}'_{\Gamma(3) \xrightarrow{i} \Gamma(2)}) \\ &= \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} (\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(3)}) \mathbf{K}'_{\Gamma(3) \xrightarrow{i} \Gamma(2)}. \quad (\because \text{Lem.47, 87, (6.2)}) \end{aligned}$$

One can see that each of  $\Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)$  and  $\Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1}$  commutes with each of  $\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)$  and  $\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1}$ , from Lem.100 and the Heisenberg commutation relations of the arguments  $\widehat{\mathbf{x}}_{\Gamma(0);j}^h$ ,  $\widehat{\mathbf{x}}_{\Gamma(0);j}^h$ ,  $\widehat{\mathbf{x}}_{\Gamma(0);i}^h$ , and  $\widehat{\mathbf{x}}_{\Gamma(0);i}^h$ . So one has

$$\mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(3)}^h \mathbf{K}_{\Gamma(3) \xrightarrow{i} \Gamma(2)}^h = \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(3)} \mathbf{K}'_{\Gamma(3) \xrightarrow{i} \Gamma(2)},$$

and hence

$$\begin{aligned} &(\mathbf{K}_{\Gamma(0) \xrightarrow{i} \Gamma(1)}^h \mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^h)^{-1} (\mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(3)}^h \mathbf{K}_{\Gamma(3) \xrightarrow{i} \Gamma(2)}^h) \\ &= (\mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)})^{-1} (\mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)})^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \\ &\quad \cdot \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(3)} \mathbf{K}'_{\Gamma(3) \xrightarrow{i} \Gamma(2)} \\ &= (\mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)})^{-1} (\mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)})^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(3)} \mathbf{K}'_{\Gamma(3) \xrightarrow{i} \Gamma(2)}, \end{aligned}$$

which we know to equal  $c_{A_1 \times A_1}^{-1} \cdot \text{id}_{\mathcal{H}_{\Gamma(2)}}$  from (6.3) of Prop.92. As each of the four operators  $(\mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)})^{-1}$ ,  $(\mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)})^{-1}$ ,  $\mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(3)}$ , and  $\mathbf{K}'_{\Gamma(3) \xrightarrow{i} \Gamma(2)}$  are special affine shift operators if we identify the Hilbert spaces by the relabelling isomorphisms (5.50), so is their composition, by Cor.74. From Lem.75 we conclude that this is the identity operator. Again, if one would like, one may directly compute  $(\mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)})^{-1} (\mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)})^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(3)} \mathbf{K}'_{\Gamma(3) \xrightarrow{i} \Gamma(2)}$  as a special affine shift operator and prove that it is the identity. ■

#### 6.5. $A_2$ identity: the ‘pentagon’ relation.

**Proposition 106.**  $c_{A_2} = 1$ .

*Proof.* Let us work with the case  $\varepsilon_{ij}^{(0)} = 1 = -\varepsilon_{ji}^{(0)}$ ; I claim that a proof for the other case is essentially the same. From (2.5) we get

$$\varepsilon_{ij}^{(1)} = -1 = -\varepsilon_{ji}^{(1)}, \quad \varepsilon_{ij}^{(2)} = 1 = -\varepsilon_{ji}^{(2)}, \quad \varepsilon_{ij}^{(3)} = -1 = -\varepsilon_{ji}^{(3)}, \quad \varepsilon_{ij}^{(4)} = -1 = -\varepsilon_{ji}^{(4)}, \quad \varepsilon_{ij}^{(5)} = 1 = -\varepsilon_{ji}^{(5)}.$$

The LHS of (6.4) is

$$\begin{aligned}
& \mathbf{K}_{\Gamma(0) \xrightarrow{i} \Gamma(1)}^{\hbar} \mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^{\hbar} \mathbf{K}_{\Gamma(2) \xrightarrow{i} \Gamma(3)}^{\hbar} \\
&= (\mathbf{K}_{\Gamma(0) \xrightarrow{i} \Gamma(1)}^{\hbar} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)}) (\mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^{\hbar} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)}) (\mathbf{K}_{\Gamma(2) \xrightarrow{i} \Gamma(3)}^{\hbar} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}) \\
&= \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \Phi^{\hbar_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^{\hbar}) \Phi^{\hbar_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^{\hbar})^{-1} \underbrace{\mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)}}_{\text{move to right}} \\
&\quad \cdot \underbrace{\Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(2);i}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(2);i}^{\hbar})^{-1} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}}_{\text{move to right}} \\
&= \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \underbrace{\mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)}}_{\text{move to right}} \Phi^{\hbar_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^{\hbar}) \Phi^{\hbar_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^{\hbar})^{-1} \\
&\quad \cdot (\Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(1);i}^{\hbar} + \underbrace{[\varepsilon_{ij}^{(1)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(1);j}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(1);i}^{\hbar} + \underbrace{[\varepsilon_{ij}^{(1)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(1);j}^{\hbar})^{-1} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)}) \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)} \quad (\cdot \text{ Lem.47, 87}) \\
&= \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \Phi^{\hbar_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar} + \underbrace{[\varepsilon_{ji}^{(0)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar} + \underbrace{[\varepsilon_{ji}^{(0)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \\
&\quad \cdot \Phi^{\hbar_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)} \quad (\cdot \text{ Lem.47, 87}) \\
&= \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar})^{-1} \Phi^{\hbar_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \\
&\quad \cdot \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}. \quad (\because \hbar_i = \hbar_j, \text{ because } d_i = d_j)
\end{aligned}$$

We would now like to move the factors  $\Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar})$  and  $\Phi^{\hbar_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})$  to the left, so that we end up in an expression of the form  $\Phi(\cdot)\Phi(\cdot)\Phi(\cdot)^{-1}\Phi(\cdot)^{-1}\Phi(\cdot)^{-1}\mathbf{K}'\mathbf{K}'\mathbf{K}'$ . To do this, we must show for example that  $\Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1}$  and  $\Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar})$  commute; this follows from Lem.100 and the Heisenberg commutation relation of the arguments  $\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar}$ . So the LHS of (6.4) becomes

$$\begin{aligned}
& \underbrace{\Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar}) \Phi^{\hbar_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})}_{\text{underlined}} \underbrace{\Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar})^{-1} \Phi^{\hbar_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1}}_{\text{underlined}} \\
& \quad \cdot \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}.
\end{aligned}$$

Note from (5.5),  $\varepsilon_{ij}^{(0)} = 1$ , and  $\hbar_j = \hbar_i$  that  $[\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}, \widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar}] = 2\pi\sqrt{-1}\hbar_i \cdot \text{id}$  on  $D_{\Gamma}$ , and that the unique self-adjoint extensions of  $\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}$  and  $\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar}$  satisfy the corresponding Weyl relations are satisfied (Lem.57). Likewise, we have  $[-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}, \widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar}] = 2\pi\sqrt{-1}\hbar_i \cdot \text{id}$ , together with the corresponding Weyl relations. Hence Cor.102 applies to the two underlined parts above, so that the LHS of (6.4) becomes

$$\begin{aligned}
& \underbrace{c_{\hbar_i} \cancel{c_{\hbar_i}'} \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar}) \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar}) \exp\left(\frac{(\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_i}\right) \exp\left(-\frac{(-\widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^2}{4\pi\sqrt{-1}\hbar_i}\right)}_{\text{underlined}} \\
& \quad \cdot \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar} - \widehat{\mathbf{x}}_{\Gamma(0);i}^{\hbar})^{-1} \Phi^{\hbar_i}(\widehat{\mathbf{x}}_{\Gamma(0);j}^{\hbar})^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}.
\end{aligned}$$

Notice the cancellation of the two  $c_{\hbar_i}$ 's. To the underbraced part we apply Lem.104, to replace it by  $\mathbf{S}_{(\mathbf{c},0)}$ , where  $\mathbf{c} = (c_{\ell m})_{\ell, m \in \{1, \dots, n\}}$  is given by

$$\mathbf{c} : \quad c_{\ell\ell} = 1, \quad \forall \ell = 1, \dots, n, \quad c_{\ell i} = -\varepsilon_{i\ell}^{(0)}, \quad \forall \ell \neq i, \quad c_{\ell m} = 0 \quad \text{otherwise.}$$

So its inverse  $\mathbf{c}^{-1} = (c^{\ell m})_{\ell, m \in \{1, \dots, n\}}$  is given by  $c^{\ell\ell} = 1, \forall \ell = 1, \dots, n, c^{\ell i} = \varepsilon_{i\ell}^{(0)}, \forall \ell \neq i, c^{\ell m} = 0$  otherwise. Hence from (5.37), (5.38), and  $\varepsilon_{ij}^{(0)} = 1$  we get

$$\begin{aligned}
\mathbf{S}_{(\mathbf{c},0)} \widehat{\mathbf{p}}_i \mathbf{S}_{(\mathbf{c},0)}^{-1} &= \widehat{\mathbf{p}}_i, & \mathbf{S}_{(\mathbf{c},0)} \widehat{\mathbf{p}}_j \mathbf{S}_{(\mathbf{c},0)}^{-1} &= \widehat{\mathbf{p}}_j + \widehat{\mathbf{p}}_i, \\
\mathbf{S}_{(\mathbf{c},0)} \widehat{\mathbf{q}}_{\ell} \mathbf{S}_{(\mathbf{c},0)}^{-1} &= \widehat{\mathbf{q}}_{\ell} + \delta_{\ell,i} \sum_{m \neq i} (-\varepsilon_{im}^{(0)}) \widehat{\mathbf{q}}_m.
\end{aligned}$$



Thus from (5.14) we have

$$\begin{aligned}
 \mathbf{S}_{(\mathbf{c},0)} \widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar} \mathbf{S}_{(\mathbf{c},0)}^{-1} &= \mathbf{S}_{(\mathbf{c},0)} \left( \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_i^{\hbar} + \sum_{\ell=1}^n \varepsilon_{i\ell}^{(0)} \widehat{\mathbf{q}}_{\ell}^{\hbar} \right) \mathbf{S}_{(\mathbf{c},0)}^{-1} = \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_i^{\hbar} + \sum_{\ell=1}^n \varepsilon_{i\ell}^{(0)} \widehat{\mathbf{q}}_{\ell}^{\hbar} = \widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar}, \\
 \mathbf{S}_{(\mathbf{c},0)} \widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar} \mathbf{S}_{(\mathbf{c},0)}^{-1} &= \mathbf{S}_{(\mathbf{c},0)} \left( \frac{d_j^{-1}}{2} \widehat{\mathbf{p}}_j^{\hbar} + \sum_{\ell=1}^n \varepsilon_{j\ell}^{(0)} \widehat{\mathbf{q}}_{\ell}^{\hbar} \right) \mathbf{S}_{(\mathbf{c},0)}^{-1} \\
 &= \frac{d_j^{-1}}{2} (\widehat{\mathbf{p}}_j^{\hbar} + \widehat{\mathbf{p}}_i^{\hbar}) + \sum_{\ell=1}^n \varepsilon_{j\ell}^{(0)} \widehat{\mathbf{q}}_{\ell}^{\hbar} - \sum_{m \neq i} (-\varepsilon_{im}^{(0)}) \widehat{\mathbf{q}}_m^{\hbar} \\
 &= \left( \frac{d_j^{-1}}{2} \widehat{\mathbf{p}}_j^{\hbar} + \sum_{\ell=1}^n \varepsilon_{j\ell}^{(0)} \widehat{\mathbf{q}}_{\ell}^{\hbar} \right) + \left( \frac{d_i^{-1}}{2} \widehat{\mathbf{p}}_i^{\hbar} + \sum_{m=1}^n \varepsilon_{im}^{(0)} \widehat{\mathbf{q}}_m^{\hbar} \right) = \widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar},
 \end{aligned}$$

as operators  $D_{\Gamma} \rightarrow D_{\Gamma}$ , where we used  $\varepsilon_{ii}^{(0)} = 0$ ,  $\varepsilon_{ji}^{(0)} = -1$ , and  $d_i = d_j$ ; such conjugation identities extend to the respective unique self-adjoint extensions. So the LHS of (6.4) becomes

$$\begin{aligned}
 &\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar}) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar}) \mathbf{S}_{(\mathbf{c},0)} \underbrace{\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar} - \widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar})^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})^{-1} \mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(1)}} \mathbf{K}'_{\Gamma^{(1)} \rightarrow \Gamma^{(2)}} \mathbf{K}'_{\Gamma^{(2)} \rightarrow \Gamma^{(3)}}}_{\substack{\text{cancel} \\ \text{cancel}}} \\
 &= \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar}) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar}) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar} + \cancel{\widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar}} - \cancel{\widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar}})^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar})^{-1} \mathbf{S}_{(\mathbf{c},0)} \\
 &\quad \cdot \mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(1)}} \mathbf{K}'_{\Gamma^{(1)} \rightarrow \Gamma^{(2)}} \mathbf{K}'_{\Gamma^{(2)} \rightarrow \Gamma^{(3)}},
 \end{aligned}$$

thanks to Lem.47.

Meanwhile, we investigate the following factor of the RHS of (6.4):

$$\begin{aligned}
 &\mathbf{K}_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}}^{\hbar} \mathbf{K}_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}}^{\hbar} = (\mathbf{K}_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}}^{\hbar} \mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}}) (\mathbf{K}_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}}^{\hbar} \mathbf{K}'_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}}) \\
 &= \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar}) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})^{-1} \mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}} \underbrace{\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(4)};i}^{\hbar}) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(4)};i}^{\hbar})^{-1} \mathbf{K}'_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}}}_{\substack{\text{cancel} \\ \text{cancel}}} \\
 &= \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar}) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar} + [\varepsilon_{ij}^{(0)}]_+ \widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar}) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar} + [\varepsilon_{ij}^{(0)}]_+ \widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})^{-1} \\
 &\quad \cdot \mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}} \mathbf{K}'_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}} \quad (\cdot \text{Lem.47, 87}) \\
 &= \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar}) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar}) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})^{-1} \mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}} \mathbf{K}'_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}},
 \end{aligned}$$

where we used  $h_i = h_j$ ,  $\varepsilon_{ij}^{(0)} = 1$ , and the fact that  $\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})^{-1}$  and  $\Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma^{(0)};i}^{\hbar} + \widehat{\mathbf{x}}_{\Gamma^{(0)};j}^{\hbar})$  commute with each other, which holds by Lem.100 and the Heisenberg commutation relation of the arguments. So, by inspection one observes that

$$\begin{aligned}
 &(\mathbf{K}_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}}^{\hbar} \mathbf{K}_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}}^{\hbar} \mathbf{P}^{(ij)})^{-1} (\mathbf{K}_{\Gamma^{(0)} \rightarrow \Gamma^{(1)}}^{\hbar} \mathbf{K}_{\Gamma^{(1)} \rightarrow \Gamma^{(2)}}^{\hbar} \mathbf{K}_{\Gamma^{(2)} \rightarrow \Gamma^{(3)}}^{\hbar}) \\
 &= \mathbf{P}^{(ij)} (\mathbf{K}'_{\Gamma^{(4)} \rightarrow \Gamma^{(5)}})^{-1} (\mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(4)}})^{-1} \mathbf{S}_{(\mathbf{c},0)} \mathbf{K}'_{\Gamma^{(0)} \rightarrow \Gamma^{(1)}} \mathbf{K}'_{\Gamma^{(1)} \rightarrow \Gamma^{(2)}} \mathbf{K}'_{\Gamma^{(2)} \rightarrow \Gamma^{(3)}}.
 \end{aligned}$$

In this last product, each of the seven factors is a special affine shift operator, hence so is the product (Cor.74). By (6.4) of Prop.93 this equals  $c_{A_2} \cdot \text{id}_{\mathcal{H}_{\Gamma^{(3)}}}$ , hence by Lem.75 we deduce that  $c_{A_2} = 1$  as desired. Again, if one would like, one may compute explicitly the composition of the above seven special affine shift operators and prove that it is the identity, without relying on Prop.93.  $\blacksquare$

## 6.6. Heuristic proofs of the hexagon and octagon identities of the non-compact quantum dilogarithm.

We shall find out soon that Fock-Goncharov's  $B_2$ -type operator identity (6.5) of Prop.94 is equivalent to the following, up to a constant and modulo some easier operator identities dealt with in §6.2.

**Conjecture 107** (the hexagon equation of the non-compact quantum dilogarithm). *Let  $P$ ,  $Q$ , and  $V$  be as in Lem.99. Then*

$$\Phi^{2\hbar}(2P) \Phi^{\hbar}(Q) = \Phi^{\hbar}(Q) \Phi^{2\hbar}(2P + 2Q) \Phi^{\hbar}(2P + Q) \Phi^{2\hbar}(2P).$$

As in the pentagon case, we will need slight a variation of this.

**Corollary 108.** *Let  $P$ ,  $Q$ , and  $V$  be as in Lem.99. Then*

$$\Phi^{2\hbar}(2P) \Phi^{\hbar}(Q) \Phi^{2\hbar}(-2P) = c_{2\hbar} \Phi^{\hbar}(Q) \Phi^{2\hbar}(2P + 2Q) \Phi^{\hbar}(2P + Q) \exp \left( \frac{(2P)^2}{4\pi\sqrt{-1}(2\hbar)} \right).$$

Instead of attempting to give a rigorous proof of Conjecture 107, here I only give a heuristic argument to ‘show’ this identity by establishing the corresponding identity for the compact quantum dilogarithm. First, let  $\mathbf{q} = e^{\pi\sqrt{-1}h}$  be a complex number of modulus strictly less than 1; then so is  $\mathbf{q}^2$  and  $1/\mathbf{q}^\vee = e^{-\pi\sqrt{-1}/(2h)}$ . We choose a square root of  $\mathbf{q}^\vee$  so that  $1/\sqrt{\mathbf{q}^\vee} = e^{-\pi\sqrt{-1}/(2h)}$ , which is also of modulus less than 1. From the infinite product expression (5.39) of the compact quantum dilogarithms one immediately observes

$$(6.10) \quad \Psi^{\mathbf{q}^2}(\mathbf{q}z) \Psi^{\mathbf{q}^2}(\mathbf{q}^{-1}z) = \Psi^{\mathbf{q}}(z),$$

and likewise  $\Psi^{1/\mathbf{q}^\vee}((\sqrt{\mathbf{q}^\vee})^{-1}z) \Psi^{1/\mathbf{q}^\vee}(\sqrt{\mathbf{q}^\vee}z) = \Psi^{1/\sqrt{\mathbf{q}^\vee}}(z)$ . We let  $\mathbf{P}$  and  $\mathbf{Q}$  be formal variables satisfying  $[\mathbf{P}, \mathbf{Q}] = 2\pi\sqrt{-1}h$ , so that by e.g. the BCH formula one has  $e^{\mathbf{P}}e^{\mathbf{Q}} = \mathbf{q}^2 e^{\mathbf{Q}}e^{\mathbf{P}}$  and  $e^{\mathbf{P}/h}e^{\mathbf{Q}/h} = (\mathbf{q}^\vee)^2 e^{\mathbf{Q}/h}e^{\mathbf{P}/h}$ ; at the moment, one may take this only heuristically.

Observe

$$\begin{aligned} & \Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})\Psi^{\mathbf{q}}(e^{\mathbf{Q}}) \\ & \stackrel{(6.10)}{=} \underline{\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})\Psi^{\mathbf{q}^2}(\mathbf{q}e^{\mathbf{Q}})}\Psi^{\mathbf{q}^2}(\mathbf{q}^{-1}e^{\mathbf{Q}}) \\ & \stackrel{(5.47)}{=} \Psi^{\mathbf{q}^2}(\mathbf{q}e^{\mathbf{Q}})\Psi^{\mathbf{q}^2}(\mathbf{q}^3e^{\mathbf{Q}}e^{2\mathbf{P}})\underline{\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})\Psi^{\mathbf{q}^2}(\mathbf{q}^{-1}e^{\mathbf{Q}})} \\ & \stackrel{(5.47)}{=} \Psi^{\mathbf{q}^2}(\mathbf{q}e^{\mathbf{Q}})\underline{\Psi^{\mathbf{q}^2}(\mathbf{q}^3e^{\mathbf{Q}}e^{2\mathbf{P}})\Psi^{\mathbf{q}^2}(\mathbf{q}^{-1}e^{\mathbf{Q}})}\Psi^{\mathbf{q}^2}(\mathbf{q}e^{\mathbf{Q}}e^{2\mathbf{P}})\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}}) \\ & \stackrel{(5.47)}{=} \underline{\Psi^{\mathbf{q}^2}(\mathbf{q}e^{\mathbf{Q}})\Psi^{\mathbf{q}^2}(\mathbf{q}^{-1}e^{\mathbf{Q}})}\Psi^{\mathbf{q}^2}(\mathbf{q}^4e^{2\mathbf{Q}}e^{2\mathbf{P}})\underline{\Psi^{\mathbf{q}^2}(\mathbf{q}^3e^{\mathbf{Q}}e^{2\mathbf{P}})\Psi^{\mathbf{q}^2}(\mathbf{q}e^{\mathbf{Q}}e^{2\mathbf{P}})}\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}}) \\ & \stackrel{(6.10)}{=} \Psi^{\mathbf{q}}(e^{\mathbf{Q}})\Psi^{\mathbf{q}^2}(e^{2\mathbf{Q}+2\mathbf{P}})\Psi^{\mathbf{q}}(e^{\mathbf{Q}+2\mathbf{P}})\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}}), \end{aligned}$$

where the pentagon equations (5.47) for  $\Psi^{\mathbf{q}^2}$  are applied to the two arguments  $e^{2\mathbf{P}}$  and  $\mathbf{q}^{\pm 1}e^{\mathbf{Q}}$  because  $(e^{2\mathbf{P}})(\mathbf{q}^{\pm 1}e^{\mathbf{Q}}) = (\mathbf{q}^2)^2(\mathbf{q}^{\pm 1}e^{\mathbf{Q}})(e^{2\mathbf{P}})$ , and to  $\mathbf{q}^3e^{\mathbf{Q}}e^{2\mathbf{P}}$  and  $\mathbf{q}^{-1}e^{\mathbf{Q}}$  because  $(\mathbf{q}^3e^{\mathbf{Q}}e^{2\mathbf{P}})(\mathbf{q}^{-1}e^{\mathbf{Q}}) = (\mathbf{q}^2)^2(\mathbf{q}^{-1}e^{\mathbf{Q}})(\mathbf{q}^3e^{\mathbf{Q}}e^{2\mathbf{P}})$ , while for the last equality we used  $e^{2\mathbf{Q}}e^{2\mathbf{P}} = \mathbf{q}^{-4}e^{2\mathbf{Q}+2\mathbf{P}}$  and  $e^{\mathbf{Q}}e^{2\mathbf{P}} = \mathbf{q}^{-2}e^{\mathbf{Q}+2\mathbf{P}}$  which are deduced from the BCH formula. So we just obtained

$$(6.11) \quad \Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})\Psi^{\mathbf{q}}(e^{\mathbf{Q}}) = \Psi^{\mathbf{q}}(e^{\mathbf{Q}})\Psi^{\mathbf{q}^2}(e^{2\mathbf{Q}+2\mathbf{P}})\Psi^{\mathbf{q}}(e^{\mathbf{Q}+2\mathbf{P}})\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}}),$$

where  $\mathbf{q} = e^{\pi\sqrt{-1}h}$  and  $[2\mathbf{P}, \mathbf{Q}] = 4\pi\sqrt{-1}h$ .

Consider the identity (6.11) when each  $\mathbf{q}$  is replaced by  $1/\sqrt{\mathbf{q}^\vee} = e^{\pi\sqrt{-1}/(-2h)}$ , while  $2\mathbf{P}$  and  $\mathbf{Q}$  are replaced respectively by  $\mathbf{Q}/h$ , and  $\mathbf{P}/h$ . This is legitimate because  $[\mathbf{Q}/h, \mathbf{P}/h] = -2\pi\sqrt{-1}h/h^2 = 4\pi\sqrt{-1}(-1/(2h))$ .

$$(6.12) \quad \Psi^{1/\mathbf{q}^\vee}(e^{\mathbf{Q}/h})\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{\mathbf{P}/h}) = \Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{\mathbf{P}/h})\Psi^{1/\mathbf{q}^\vee}(e^{(2\mathbf{P}+\mathbf{Q})/h})\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{(\mathbf{P}+\mathbf{Q})/h})\Psi^{1/\mathbf{q}^\vee}(e^{\mathbf{Q}/h}).$$

Meanwhile, it is natural to regard that each of  $e^{2\mathbf{P}}$  and  $e^{\mathbf{Q}}$  commutes with each of  $e^{\mathbf{Q}/h}$  and  $e^{\mathbf{P}/h}$ , e.g. from the point of view of the BCH formula, for

$$[2\mathbf{P}, \mathbf{Q}/h] = 4\pi\sqrt{-1}, \quad [2\mathbf{P}, \mathbf{P}/h] = 0, \quad [\mathbf{Q}, \mathbf{Q}/h] = 0, \quad [\mathbf{Q}, \mathbf{P}/h] = -2\pi\sqrt{-1};$$

further, we assume that any continuous function in each of  $e^{2\mathbf{P}}$  and  $e^{\mathbf{Q}}$  commutes with any continuous function in each of  $e^{\mathbf{Q}/h}$  and  $e^{\mathbf{P}/h}$ . This is only heuristic, and perhaps the functional calculus for explicit operators is a way to make such an argument rigorous; however, let us not bother doing this here. Also, since  $\mathbf{q}^2 = e^{\pi\sqrt{-1}(2h)}$ , we have  $(\mathbf{q}^2)^\vee = e^{\pi\sqrt{-1}/(2h)} = \sqrt{\mathbf{q}^\vee}$ . So

$$\begin{aligned} \Phi^{2h}(2\mathbf{P})\Phi^h(\mathbf{Q}) & \stackrel{(5.43)}{=} \Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})(\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{\mathbf{P}/h}))^{-1}\Psi^{\mathbf{q}}(e^{\mathbf{Q}})(\Psi^{1/\mathbf{q}^\vee}(e^{\mathbf{Q}/h}))^{-1} \\ & \stackrel{\vee}{=} \underline{\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})\Psi^{\mathbf{q}}(e^{\mathbf{Q}})}(\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{\mathbf{P}/h}))^{-1}(\Psi^{1/\mathbf{q}^\vee}(e^{\mathbf{Q}/h}))^{-1}, \end{aligned}$$

where for the checked equality we used the commutation relations. Meanwhile,

$$\begin{aligned} & \Phi^h(\mathbf{Q})\Phi^{2h}(2\mathbf{P}+2\mathbf{Q})\Phi^h(2\mathbf{P}+\mathbf{Q})\Phi^{2h}(2\mathbf{P}) \\ & \stackrel{(5.43)}{=} \Psi^{\mathbf{q}}(e^{\mathbf{Q}})(\Psi^{1/\mathbf{q}^\vee}(e^{\mathbf{Q}/h}))^{-1}\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}+2\mathbf{Q}})(\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{(\mathbf{P}+\mathbf{Q})/h}))^{-1} \\ & \quad \cdot \Psi^{\mathbf{q}}(e^{2\mathbf{P}+\mathbf{Q}})(\Psi^{1/\mathbf{q}^\vee}(e^{(\mathbf{P}+\mathbf{Q})/h}))^{-1}\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})(\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{\mathbf{P}/h}))^{-1} \\ & \stackrel{\vee}{=} \underline{\Psi^{\mathbf{q}}(e^{\mathbf{Q}})\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}+2\mathbf{Q}})\Psi^{\mathbf{q}}(e^{2\mathbf{P}+\mathbf{Q}})\Psi^{\mathbf{q}^2}(e^{2\mathbf{P}})} \\ & \quad \cdot (\Psi^{1/\mathbf{q}^\vee}(e^{\mathbf{Q}/h}))^{-1}(\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{(\mathbf{P}+\mathbf{Q})/h}))^{-1}(\Psi^{1/\mathbf{q}^\vee}(e^{(\mathbf{P}+\mathbf{Q})/h}))^{-1}(\Psi^{1/\sqrt{\mathbf{q}^\vee}}(e^{\mathbf{P}/h}))^{-1}, \end{aligned}$$

where for the checked equality we used the commutation relations. A reader can immediately verify by inspection that the application of (6.11) and (6.12) on the underlined parts yields  $\Phi^{2h}(2\mathbf{P})\Phi^h(\mathbf{Q}) = \Phi^h(\mathbf{Q})\Phi^{2h}(2\mathbf{P} + 2\mathbf{Q})\Phi^h(2\mathbf{P} + \mathbf{Q})\Phi^{2h}(2\mathbf{P})$ . We then (heuristically) take a ‘limit’ as  $\text{Im}(h) \rightarrow 0$  to obtained the desired result.

Likewise, one may also obtain a heuristic proof of the identity corresponding to Fock-Goncharov’s  $G_2$ -type identity (6.6) of Prop.95. This heuristic proof of the  $G_2$  identity can be found in [KY]; here let me give a sketch. One first observes  $\Psi^{\mathbf{q}^3}(\mathbf{q}^{-2}z)\Psi^{\mathbf{q}^3}(z)\Psi^{\mathbf{q}^3}(\mathbf{q}^2z) = \Psi^{\mathbf{q}}(z)$ ; using this and the pentagon equation for  $\Psi^{\mathbf{q}^3}$  one can show

$$\Psi^{\mathbf{q}^3}(e^{3\mathbf{P}})\Psi^{\mathbf{q}}(e^{\mathbf{Q}}) = \Psi^{\mathbf{q}}(e^{\mathbf{Q}})\Psi^{\mathbf{q}^3}(e^{3\mathbf{Q}+3\mathbf{P}})\Psi^{\mathbf{q}}(e^{2\mathbf{Q}+3\mathbf{P}})\Psi^{\mathbf{q}^3}(e^{3\mathbf{Q}+6\mathbf{P}})\Psi^{\mathbf{q}}(e^{\mathbf{Q}+3\mathbf{P}})\Psi^{\mathbf{q}^3}(e^{3\mathbf{P}})$$

where  $[\mathbf{P}, \mathbf{Q}] = 2\pi\sqrt{-1}h$ , which suggests

**Conjecture 109** (the octagon equation of the non-compact quantum dilogarithm). *Let  $P$ ,  $Q$ , and  $V$  be as in Lem.99. Then*

$$\Phi^{3h}(3P)\Phi^h(Q) = \Phi^h(Q)\Phi^{3h}(3P+3Q)\Phi^h(3P+2Q)\Phi^{3h}(6P+3Q)\Phi^h(3P+Q)\Phi^{3h}(3P).$$

Ivan Ip told me that the two identities of unitary operators in the above Conjectures 107 and 109 would follow from certain modification of the operator identities that he established in [I15, §3]; I myself was not yet able to obtain a rigorous proof this way.

### 6.7. $B_2$ identity: the ‘hexagon’ relation.

**Proposition 110.**  $c_{B_2} = 1$ .

*Proof.* Let us work with the case  $\varepsilon_{ij}^{(0)} = 2$ ,  $\varepsilon_{ji}^{(0)} = -1$ ; I claim that a proof for the other case is essentially the same. From (2.5) we get

$$(6.13) \quad \varepsilon_{ij}^{(1)} = -\varepsilon_{ij}^{(0)} = -2, \quad \varepsilon_{ji}^{(4)} = -\varepsilon_{ji}^{(0)} = 1,$$

The LHS of (6.5) is

$$\begin{aligned} & \mathbf{K}_{\Gamma(0) \xrightarrow{i} \Gamma(1)}^h \mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^h \mathbf{K}_{\Gamma(2) \xrightarrow{i} \Gamma(3)}^h \\ &= (\mathbf{K}_{\Gamma(0) \xrightarrow{i} \Gamma(1)}^{\#h} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)}) (\mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^{\#h} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)}) (\mathbf{K}_{\Gamma(2) \xrightarrow{i} \Gamma(3)}^{\#h} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}) \\ &= \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^h)^{-1} \underbrace{\mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)}}_{\text{move to right}} \\ & \quad \cdot \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(2);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(2);i}^h)^{-1} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)} \\ &= \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \underbrace{\mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)}}_{\text{move to right}} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(1);j}^h)^{-1} \\ & \quad \cdot \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(1);i}^h + \underbrace{[\varepsilon_{ij}^{(1)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(1);j}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(1);i}^h + \underbrace{[\varepsilon_{ij}^{(1)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(1);j}^h)^{-1} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)} \quad (\because \text{Lem.47, 87}) \\ &= \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h + \underbrace{[\varepsilon_{ji}^{(0)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h + \underbrace{[\varepsilon_{ji}^{(0)}]_+}_{=0} \widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \\ & \quad \cdot \Phi^{h_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)} \quad (\because \text{Lem.47, 87}). \end{aligned}$$

Like in the proof of Prop.106, we use Lem.100 (and Heisenberg commutation relations) to move some factors around, so that the LHS of (6.5) becomes

$$\begin{aligned} & \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^{-1} \\ & \quad \cdot \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}. \end{aligned}$$

From  $\varepsilon_{ij}^{(0)} = 2$  and  $\varepsilon_{ji}^{(0)} = -1$  note that  $d_j = 2d_i$ , so  $h_i = 2h_j$ . As  $[\widehat{\mathbf{x}}_{\Gamma(0);i}^h, \widehat{\mathbf{x}}_{\Gamma(0);j}^h] \stackrel{(5.5)}{=} 2\pi\sqrt{-1}h_j\varepsilon_{ij}^{(0)} \cdot \text{id} = 4\pi\sqrt{-1}h_j \cdot \text{id} = [-\widehat{\mathbf{x}}_{\Gamma(0);i}^h, \widehat{\mathbf{x}}_{\Gamma(0);j}^h]$ , one may apply Cor.108 to the two underlined parts above, so that the LHS of

(6.5) becomes

$$\begin{aligned} & c_{2h_j}^{-1} c_{2h_j}^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + 2\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + \widehat{\mathbf{x}}_{\Gamma(0);j}^h) \exp\left(\frac{(\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^2}{4\pi\sqrt{-1}h_i}\right) \exp\left(-\frac{(-\widehat{\mathbf{x}}_{\Gamma(0);i}^h)^2}{4\pi\sqrt{-1}h_i}\right) \\ & \cdot \Phi^{h_j}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^h + \widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_i}(-\widehat{\mathbf{x}}_{\Gamma(0);i}^h + 2\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}, \end{aligned}$$

where now Lem.104 lets us to replace the underbraced part by  $\mathbf{S}_{(\mathbf{c},0)}$ , where  $\mathbf{c} = (c_{\ell m})_{\ell,m \in \{1,\dots,n\}}$  is given by

$$\mathbf{c} : \quad c_{\ell\ell} = 1, \quad \forall \ell = 1, \dots, n, \quad c_{\ell i} = -\varepsilon_{i\ell}^{(0)}, \quad \forall \ell \neq i, \quad c_{\ell m} = 0 \quad \text{otherwise.}$$

In particular, note the cancellation of the two constants  $c_{2h_j}$  and  $c_{2h_j}^{-1}$ . Note that this  $\mathbf{c}$  is the same as the one appearing in the proof of Prop.106, except that we now have  $d_j = 2d_i$  and  $\varepsilon_{ij}^{(0)} = 2$ . One can carefully verify that we still have

$$\mathbf{S}_{(\mathbf{c},0)} \widehat{\mathbf{x}}_{\Gamma(0);i}^h \mathbf{S}_{(\mathbf{c},0)}^{-1} = \widehat{\mathbf{x}}_{\Gamma(0);i}^h, \quad \mathbf{S}_{(\mathbf{c},0)} \widehat{\mathbf{x}}_{\Gamma(0);j}^h \mathbf{S}_{(\mathbf{c},0)}^{-1} = \widehat{\mathbf{x}}_{\Gamma(0);j}^h + \widehat{\mathbf{x}}_{\Gamma(0);i}^h.$$

So, by moving  $\mathbf{S}_{(\mathbf{c},0)}$  to the right with the help of Lem.47, the LHS of (6.5) becomes

$$\begin{aligned} & \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + 2\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + \widehat{\mathbf{x}}_{\Gamma(0);j}^h) \\ & \cdot \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + 2\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + \widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{S}_{(\mathbf{c},0)} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}. \end{aligned}$$

Meanwhile, the RHS of (6.5) without the constant is

$$\begin{aligned} & \mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(4)}^h \mathbf{K}_{\Gamma(4) \xrightarrow{i} \Gamma(5)}^h \mathbf{K}_{\Gamma(5) \xrightarrow{j} \Gamma(3)}^h \\ & = (\mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(4)}^{\#h} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(4)}) (\mathbf{K}_{\Gamma(4) \xrightarrow{i} \Gamma(5)}^{\#h} \mathbf{K}'_{\Gamma(4) \xrightarrow{i} \Gamma(5)}) (\mathbf{K}_{\Gamma(5) \xrightarrow{j} \Gamma(3)}^{\#h} \mathbf{K}'_{\Gamma(5) \xrightarrow{j} \Gamma(3)}) \\ & = \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(4)} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(4);i}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(4);i}^h)^{-1} \underbrace{\mathbf{K}'_{\Gamma(4) \xrightarrow{i} \Gamma(5)}}_{\text{move to right}} \\ & \cdot \frac{\Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(5);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(5);j}^h)^{-1} \mathbf{K}'_{\Gamma(5) \xrightarrow{j} \Gamma(3)}}{\Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(4)}} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(4);i}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(4);i}^h)^{-1} \\ & \cdot \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(4);j}^h + [\varepsilon_{ji}^{(4)}]_+ \widehat{\mathbf{x}}_{\Gamma(4);i}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(4);j}^h + [\varepsilon_{ji}^{(4)}]_+ \widehat{\mathbf{x}}_{\Gamma(4);i}^h)^{-1} \mathbf{K}'_{\Gamma(4) \xrightarrow{i} \Gamma(5)} \mathbf{K}'_{\Gamma(5) \xrightarrow{j} \Gamma(3)} \quad (\because \text{Lem.47, 87}) \\ & = \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + [\varepsilon_{ij}^{(0)}]_+ \widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + [\varepsilon_{ij}^{(0)}]_+ \widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \\ & \cdot \Phi^{h_j}(-\widehat{\mathbf{x}}_{\Gamma(0);j}^h + \widehat{\mathbf{x}}_{\Gamma(0);i}^h + [\varepsilon_{ij}^{(0)}]_+ \widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(-\widehat{\mathbf{x}}_{\Gamma(0);j}^h + \widehat{\mathbf{x}}_{\Gamma(0);i}^h + [\varepsilon_{ij}^{(0)}]_+ \widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \\ & \cdot \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(4)} \mathbf{K}'_{\Gamma(4) \xrightarrow{i} \Gamma(5)} \mathbf{K}'_{\Gamma(5) \xrightarrow{j} \Gamma(3)} \quad (\because \text{Lem.47, 87}) \\ & = \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + 2\widehat{\mathbf{x}}_{\Gamma(0);j}^h) \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + \widehat{\mathbf{x}}_{\Gamma(0);j}^h) \\ & \cdot \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_i}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + 2\widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \Phi^{h_j}(\widehat{\mathbf{x}}_{\Gamma(0);i}^h + \widehat{\mathbf{x}}_{\Gamma(0);j}^h)^{-1} \mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(4)} \mathbf{K}'_{\Gamma(4) \xrightarrow{i} \Gamma(5)} \mathbf{K}'_{\Gamma(5) \xrightarrow{j} \Gamma(3)}, \end{aligned}$$

where we used  $\varepsilon_{ij}^{(0)} = 2$ , and the commutation relations coming from Lem.100 (and Heisenberg commutation relations) to move some factors around. Thus, by inspection one has

$$\begin{aligned} & (\mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(4)}^h \mathbf{K}_{\Gamma(4) \xrightarrow{i} \Gamma(5)}^h \mathbf{K}_{\Gamma(5) \xrightarrow{j} \Gamma(3)}^h)^{-1} \mathbf{K}_{\Gamma(0) \xrightarrow{j} \Gamma(1)}^h \mathbf{K}_{\Gamma(1) \xrightarrow{j} \Gamma(2)}^h \mathbf{K}_{\Gamma(2) \xrightarrow{i} \Gamma(3)}^h \\ & = (\mathbf{K}'_{\Gamma(5) \xrightarrow{j} \Gamma(3)})^{-1} (\mathbf{K}'_{\Gamma(4) \xrightarrow{i} \Gamma(5)})^{-1} (\mathbf{K}'_{\Gamma(0) \xrightarrow{j} \Gamma(4)})^{-1} \mathbf{S}_{(\mathbf{c},0)} \mathbf{K}'_{\Gamma(0) \xrightarrow{i} \Gamma(1)} \mathbf{K}'_{\Gamma(1) \xrightarrow{j} \Gamma(2)} \mathbf{K}'_{\Gamma(2) \xrightarrow{i} \Gamma(3)}. \end{aligned}$$

Eq.(6.5) of Prop.94 tells us that this equals  $c_{B_2} \cdot \text{id}_{\mathcal{H}_{\Gamma(3)}}$ , while it is a special affine shift operator, because it is a composition of special affine shift operators. Hence Lem.75 says  $c_{B_2} = 1$  as desired. Again, this last step can be done directly without resorting to Prop.94.  $\blacksquare$

## 6.8. $G_2$ identity: the ‘octagon’ relation.

**Proposition 111.**  $c_{G_2} = 1$ .

I think I presented enough detail of the proof of the other cases, so that a reader can easily construct a proof of this last proposition along a similar line. In particular, it is quite natural to expect that the constants coming from operator identities will cancel each other.

## REFERENCES

- [B01] E. W. Barnes, *Theory of the double gamma function*, Phil. Trans. Roy. Soc. **A196** (1901) 265–388.
- [BZ05] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Adv. Math. **195**(2) (2005) 405–455. arXiv:math.QA/0404446
- [F95] L. D. Faddeev, *Discrete Heisenberg-Weyl group and modular group*, Lett. Math. Phys. **34** (1995) 249–254
- [FK94] L. D. Faddeev and R. M. Kashaev, *Quantum dilogarithm*, Modern Phys. Lett. **A9** (1994) 427–434
- [FKV01] L. D. Faddeev, R. M. Kashaev, and A. Y. Volkov, *Strongly coupled quantum discrete Liouville theory, I: Algebraic approach and duality*, Commun. Math. Phys. **219** (2001) 199–219
- [FG07] V. V. Fock and A. B. Goncharov, “Cluster ensembles, quantization and the dilogarithm II: The intertwiner” in *Manin’s Festschrift*, Birkhäuser, 2007. arXiv:math.QA/0702398
- [FG09] V. V. Fock and A. B. Goncharov, *The quantum dilogarithm and representations of the quantum cluster varieties*, Inventiones Math. **175** (2009) 223–286.
- [FG09b] V. V. Fock and A. B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér. (4) **42**(6) (2009) 865–930. arXiv:math.AG/0311245
- [FZ02] S. Fomin and A. Zelevinsky, *Cluster Algebras I: Foundations*, J. Am. Math. Soc. **15**(2) (2002) 497–529.
- [FZ07] S. Fomin and A. Zelevinsky, *Cluster Algebras IV: Coefficients*, Compositio Math. **143** (2007) 112–164.
- [FK12] I. B. Frenkel and H. Kim, *Quantum Teichmüller space from the quantum plane*, Duke Math. J. **161** no.2 (2012) 305–366.
- [FS10] L. Funar and V. Sergiescu, *Central extensions of the Ptolemy-Thompson group and quantized Teichmüller theory*, Journal of Topology **3** (2010) 29–62.
- [G08] A. B. Goncharov, “Pentagon relation for the quantum dilogarithm and quantized  $\mathcal{M}_{0,5}^{\text{cyc}}$ ” in *Geometry and Dynamics of Groups and Spaces*, Progr. Math. **265**, Birkhäuser, Basel, 2008, pp.415–428.
- [I15] I. C. Ip, *Positive representations of split real quantum groups: the universal R operator*, Int. Math. Res. Not. IMRN 2015 no.1, 240–287. arXiv:1212.5149
- [KN11] R. M. Kashaev and T. Nakanishi, *Classical and quantum dilogarithm identities*, Symmetry, Integrability and Geometry: Methods and Applications SIGMA **7** (2011), 102, 29 pages. arXiv:1104.4630
- [K12] H. Kim, *The dilogarithmic central extension of the Ptolemy-Thompson group via the Kashaev quantization*, to appear in Adv. Math. arXiv:1211.4300
- [K16] H. Kim, *Phase constants in the Fock-Goncharov quantization of cluster varieties: long version*, arXiv:1602.00361
- [KS] H. Kim and C. Scarinci, *Quantization of moduli spaces of three-dimensional spacetimes* (tentative), in preparation.
- [KY] H. Kim and M. Yamazaki, *Cluster groupoid for (non)-finite type quivers* (tentative), in preparation
- [K31] B. O. Koopman, *Hamiltonian systems and transformations in Hilbert space*, Proc. Natl. Acad. Sci. **17** (1931) 315–318.
- [RS80] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics I: Functional Analysis.*, Revised and enlarged ed., Academic Press, San Diego, 1980
- [S32] M. H. Stone, *On One-Parameter Unitary Groups in Hilbert Space*, Ann. Math. **33** no.3 (1932) 643–648.
- [v31] J. von Neumann, *Die Eindeutigkeit der Schrödingerschen Operatoren*, Mathematische Annalen **104** (1931) 570–578. (in German)
- [W00] S. L. Woronowicz, *Quantum exponential function*, Rev. Math. Phys. **12** no.6 (2000) 873–920.
- [X14] B. Xu, *Central extension of mapping class group via Chekhov-Fock quantization*, arXiv:1410.5551
- [Y80] Kōsaku Yosida, *Functional Analysis*, 6th ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **123**, Springer-Verlag, Berlin-New York, 1980.

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